Chordal structure and polynomial systems

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Joint work with **Diego Cifuentes** (MIT)
`arXiv:1411.1745`, `arXiv:1507.03046`

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Polynomial systems and graphs

A polynomial system defined by m equations in n variables:

$$f_i(x_0, \ldots, x_{n-1}) = 0, \qquad i = 1, \ldots, m$$

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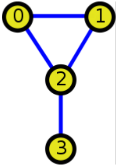
Example:

$$I = \langle x_0^2 x_1 x_2 + 2x_1 + 1, \quad x_1^2 + x_2, \quad x_1 + x_2, \quad x_2 x_3 \rangle$$



Questions

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- Can the graph structure help *solve* this system?
- For instance, to optimize, or to compute Groebner bases?
- Or, perhaps we can do something *better*?
- Preserve graph (sparsity) structure?
- Complexity aspects?

(Hyper)Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

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Reasonably well-known in discrete (0/1) optimization, what happens in the continuous side?
(e.g., Waki et al., Lasserre, Bienstock, Jordan/Wainwright, Lavaei, etc)

Chordality, treewidth, and a meta-theorem

Let G be a graph with vertices $x_0, \ldots, x_{n-1}$.
A vertex ordering $x_0 > x_1 > \cdots > x_{n-1}$ is a *perfect elimination ordering* if for each x_l, the set

$$X_l := \{x_l\} \cup \{x_m : x_m \text{ is adjacent to } x_l,\ \ x_l > x_m\}$$

is such that the restriction $G|_{X_l}$ is a clique.
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> Meta-theorem:
> NP-complete problems are "easy" on graphs of small treewidth.

Bad news? (I)

Subset sum problem, with data $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}$.
Is there a subset of A that adds up to S?

Letting s_i be the partial sums, we can write a polynomial system:

$$\begin{aligned} 0 &= s_0 \\ 0 &= (s_i - s_{i-1})(s_i - s_{i-1} - a_i) \\ S &= s_n \end{aligned}$$

The graph associated with these equations is a path (treewidth=1)

But, subset sum is NP-complete... :(

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For polynomials, however, Groebner bases can destroy chordality.

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Every Groebner basis must contain the polynomial $x_0 - x_1$, breaking the sparsity structure.

Two papers

- Chordal elimination and Groebner bases (arXiv:1411:1745)
  - New *chordal elimination* algorithm, to exploit graphical structure.
  - Conditions under which chordal elimination succeeds.
  - For a certain class, complexity is *linear* in number of variables! (exponential in treewidth)
  - Implementation and experimental results
- Computing permanents, hyperdeterminants, and mixed discriminants (arXiv:1507:03046)
  - New polynomial time algorithm $O(n2^{\omega})$ (ω is treewidth).
  - Hardness: mixed volume still hard, even with small treewidth.

Chordal elimination (sketch)

Given equations, construct graph G, a chordal completion, and a perfect elimination ordering.

Will produce a decreasing sequence of ideals $I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{n-1}$.

Given current ideal I_l, split the generators

$$I_l = \underbrace{J_l}_{\in \mathbb{K}[X_l]} + \underbrace{K_{l+1}}_{\notin \mathbb{K}[X_l]}$$

and eliminate variable x_l

$$I_{l+1} = \mathrm{elim}_{l+1}(J_l) + K_{l+1}$$

"Ideally" (!), I_l should be the l-th elimination ideal $\mathrm{elim}_l(I)$...

Notice that by chordality, graph structure is always preserved!

When does chordal elimination succeed?

We need conditions for this to work, i.e., for $\mathbf{V}(I_l) = \mathbf{V}(\mathrm{elim}_l(I))$.

Thm 1: Let I be an ideal and assume that for each l such that X_l is a maximal clique of G, the ideal $J_l \subseteq \mathbb{K}[X_l]$ is zero dimensional. Then, chordal elimination succeeds.

In particular, finite fields $\mathbb{F}_q$, and 0/1 problems.

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Def: A polynomial f is *simplicial* if for each variable x_l, the monomial m_l of largest degree in x_l is unique and has the form $m_l = x^{d_l}$.

Thm 2: Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal such that for each $1 \leq i \leq s$, f_i is generic simplicial. Then, chordal elimination succeeds.

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[Intuition: interaction of (iterated) "closure/extension thm" + chordality]
[Intuition: variety has "small" coordinate projections, can compute those, and glue them]

Complexity

For "nice" cases, complexity is *linear* in number of variables n, number of equations s, and exponential in treewidth κ.

Thm: Let I be such that each (maximal) $\tilde{H}^j$ is q-dominated. The complexity of computing I_l is $\tilde{O}(s + lq^{\alpha\kappa})$. We can find all elimination ideals in $\tilde{O}(nq^{\alpha\kappa})$.

E.g., we recover known results on linear-time colorability for bounded treewidth:
Cor: Let G be a graph and $\bar{G}$ a chordal completion with largest clique of size κ. We can describe all q-colorings of G in $\tilde{O}(nq^{\alpha\kappa})$.

Implementation and examples

Implemented in Sage, using Singular and PolyBoRi (for $\mathbb{F}_2$).

- Graph colorings (counting q-colorings)
- Cryptography ("baby" AES, Cid *et al.*)
- Sensor Network localization
- Discretization of polynomial equations

Results: Crypto - AES variant (Cid et al.) - $\mathbb{F}_2[x]$

Performance on $SR(n, 1, 2, 4)$ for chordal elimination, and computing (lex/degrevlex) Gröbner bases (PolyBoRi).

n	Variables	Equations	Seed	ChordElim	LexGB	DegrevlexGB
6	176	320	0	575.516	402.255	256.253
			1	609.529	284.216	144.316
			2	649.408	258.965	133.367
10	288	528	0	941.068	$>$ 1100, aborted	1279.879
			1	784.709	$>$ 1400, aborted	1150.332
			2	1124.942	$>$ 3600, aborted	$>$ 2500, aborted

- For small problems standard Gröbner bases outperform chordal elimination, particularly using degrevlex order.
- Nevertheless, chordal elimination scales better, being faster than both methods for $n = 10$.
- In addition, standard Gröbner bases have higher memory requirements, which is reflected in the many experiments that aborted for this reason.

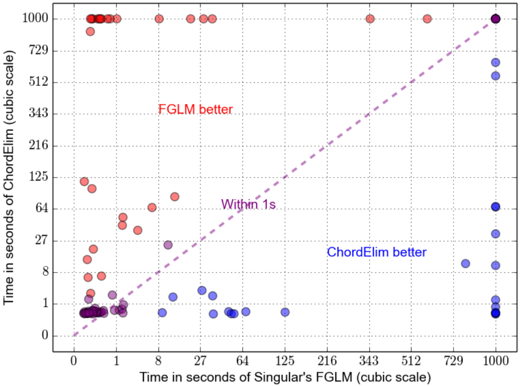
Results: Sensor network localization - $\mathbb{Q}[x]$

Find positions, given a few known fixed anchors and pairwise distances.
Comparison with Singular: DegrevlexGB, LexFGLM

- Natural graph structure

$$\|x_i - x_j\|^2 = d_{ij}^2 \quad ij \in \mathcal{A}$$
$$\|x_i - a_k\|^2 = e_{ij}^2 \quad ik \in \mathcal{B}$$

- Simplicial, therefore exact elimination
- Underconstrained regime: chordal is much better
- Overconstrained regime: competitive (plot)



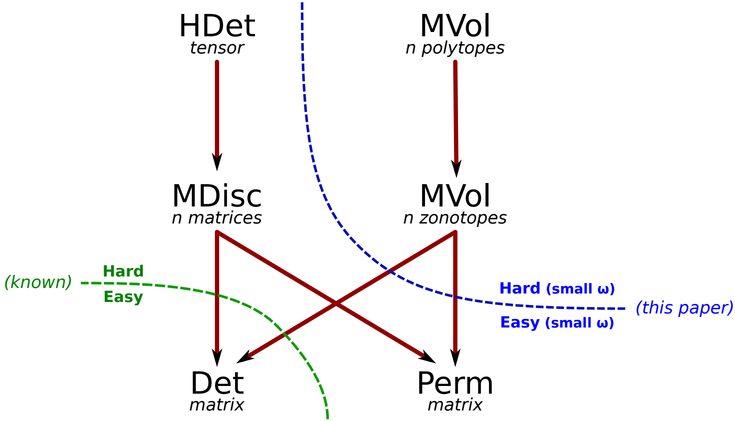
Permanents, . . .

- The *permanent* of a matrix is

$$\mathrm{perm}(M) := \sum_{\pi} \prod_{i=1}^{n} M_{i,\pi(i)}$$

  where the sum is over all permutations $\pi \in S_n$.
- Very difficult (#P-hard).
- What happens under small treewidth?
- What about generalizations (e.g., mixed discriminants, mixed volumes, etc)?

An image is worth more than...



- New tree-decomposition (DP) algorithms for permanents, mixed discriminants and hyperdeterminants
- Hardness results for mixed volumes and above.

Summary

- (Hyper)graphical structure *may* simplify optimization/solving
- Under assumptions (treewidth + algebraic structure), tractable!
- Yields practical, competitive, implementable algorithms
- Ongoing and future work: other polynomial solving approaches (e.g., triangular sets, homotopies, full numerical algebraic geometry...)

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If you want to know more:

- D. Cifuentes, P.A. Parrilo, Exploiting chordal structure in polynomial ideals: a Groebner basis approach. `arXiv:1411.1745`.
- D. Cifuentes, P.A. Parrilo, An efficient tree decomposition method for permanents and mixed discriminants `arXiv:1507.03046`.

Thanks for your attention!