Photogrammetry Profiles and Classical Invariant Theory

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Quotient Varieties

Let n, d be positive integers. The group $\mathrm{Aut}(\mathbb{P}^d) = \mathrm{PGL}(d+1)$ acts on $(\mathbb{P}^d)^n$ by

$$(p_1, \dots, p_n)^\sigma := (p_1^\sigma, \dots, p_n^\sigma)$$

Is the quotient set an algebraic variety?
(not quite, but almost)

The Invariant Ring

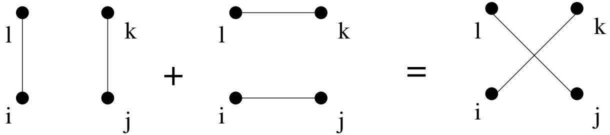
The group $\mathrm{SE}_3(d+1)$ acts on $R := \mathbb{C}[p_{1,1}, \ldots, p_{n,d+1}]$ by substitution.
The invariant ring contains the "brackets", defined as

$$[i_1 i_2 \ldots i_{d+1}] := \det(p_{i_1}, \ldots, p_{i_{d+1}}), \; i_1 < \cdots < i_{d+1}$$

If $d = 1$, then the brackets generate the invariant ring. They fulfill the Plücker relations

$$[ij][kl] + [il][jk] = [ik][jl], \; i < j < k < l$$

Graphical Representation of Plücker Relation



A bracket polynomial $[ij]$ is represented by an edge in a planar graph.
A product of bracket polynomials is represented by a graph (undirected, multiple edges allowed, no loops).

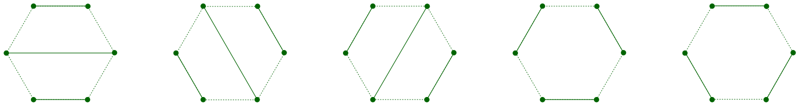
The Quotient Variety M_4^1

Generators of degree $(1,1,1,1)$ are: $x_1 := [12][34], x_2 := [14][23]$.

The quotient variety is isomorphic to $\mathbb{P}^1$. The quotient $x_1 : x_2$ is known as *cross ratio*.

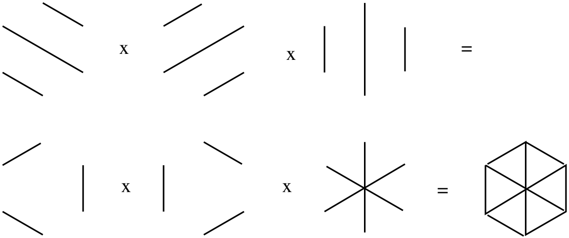
The Quotient Variety M_6^1

Generators of degree $(1,1,1,1,1,1)$ correspond to triples of disjoint edges:



[12][36][45]	[16][25][34]	[14][23][56]	[12][34][56]	[16][23][45]
x_1	x_2	x_3	x_4	x_5

The Equation of M_6^1



For the last summand in the second line, we get

$$[14][25][36] = -x_1 - x_2 - x_3 - x_4 - x_5$$

by the Plücker relations. Hence the equation of M_6^1 is

$$x_1 x_2 x_3 + x_4 x_5 (x_1 + x_2 + x_3 + x_4 + x_5) = 0.$$

Geometry of M_6^1

M_6^1 is the Segre cubic hypersurface S_3; it can be characterized as the one with a maximal number of nodes (10).

S_3 has 15 planes E_{ij}, $1 \leq i < j \leq 6$, corresponding to the set of 6-tuples such that $p_i = p_j$.

The Quotient Variety M_6^2

Here $\mathrm{PGL}(3)$ acts on $(\mathbb{P}^2)^6$. The invariants of degree $(1,1,1,1,1,1)$ are generated by

$$x_1 := [123][456], x_2 := [124][356], x_3 := [125][346],$$

$$x_4 := [134][256], x_5 := [135][246],$$

and they are algebraically independent.

The graded invariant ring is generated by x_1, x_2, x_3, x_4, x_5 and

$$x_6 := \det(p_1^{\otimes 2}, \ldots, p_6^{\otimes 2}).$$

There is a single algebraic equation of the form

$$x_6^2 - F_4(x_1, x_2, x_3, x_4, x_5).$$

Geometry of M_6^2

$\gamma : M_6^2 \to \mathbb{P}^4$ is a double cover branched over the quartic hypersurface defined by F_4: the Igusa quartic. Its points corresponds to 6-tuples lying on a conic.

The Igusa quartic is the dual of the Segre cubic, i.e., we have a correspondence

$$\begin{aligned} \text{points in } S_3 \quad &\leftrightarrow \text{tangent hyperplanes to } F_4 \\ \text{points in } F_4 \quad &\leftrightarrow \text{tangent hyperplanes to } S_3 \end{aligned}$$

Gale Duality

For any $d_1, d_2 \geq 1$, there is an isomorphism $M^{d_1-1}_{d_1+d_2} \cong M^{d_2-1}_{d_1+d_2}$:

$$\begin{bmatrix} (1: & \cdots & :0), \\ \cdots & \cdots & \cdots \\ (0: & \cdots & :1), \\ (a_{1,1}: & \cdots & :a_{1,d_1}), \\ \cdots & \cdots & \cdots \\ (a_{d_2,1}: & \cdots & :a_{d_2,d_1}) \end{bmatrix} \cong \begin{bmatrix} (1: & \cdots & :0), \\ \cdots & \cdots & \cdots \\ (0: & \cdots & :1), \\ (a_{1,1}: & \cdots & :a_{d_2,1}), \\ \cdots & \cdots & \cdots \\ (a_{1,d_2}: & \cdots & :a_{d_2,d_1}) \end{bmatrix}$$

In particular, the automorphism $M^2_6 \cong M^2_6$ interchanges the sheets of the double cover.

Note also that $M^1_6 \cong M^3_6 \cong S_3$.

1D Pictures of 6 Points in the Plane

The photographic map

$$\phi : (\mathbb{P}_2)^6 \times \mathbb{P}^2 \dashrightarrow S_3,$$

$$((p_1, \ldots, p_6), p_0) \mapsto [\pi_{p_0}(p_1, \ldots, p_6)]_{\mathrm{PGL}(2)}$$

assigns to each object of 6 points in the plane and camera position in the plane a picture of 6 points in the line, up to equivalence. Explicitly, it is given by

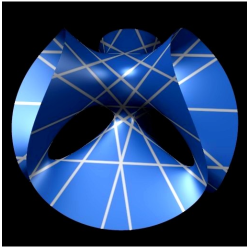
$$(x_1 : x_2 : x_3 : x_4 : x_5) = ([012][036][045] : [016][025][034]$$

$$: [014][023][056] : [012][034][056] : [016][023][045]).$$

The subvariety $\phi((p_1, \times, p_6) \times \mathbb{P}^2) \subset S_3$ is called the *profile* of $(p_1, \ldots, p_6)$.

Geometry of Photographic Map and Profile

If the object $O := (p_1, \dots, p_6)$ is fixed, then $\phi_O : \mathbb{P}^2 \dashrightarrow S_3$ is given by cubic forms vanishing at $p_1, \dots, p_6$.



The profile $\phi_O(\mathbb{P}^2)$ is a hyperplane section of S_3; in general, it is a nonsingular cubic surface.
The profile has 27 lines. 15 are intersections of the planes E_{ij} with the hyperplane. The remaining 12 form a *Schläfli double-six*.

Photogrammetry

We consider the problem of recovering $[(p_1, \ldots, p_6)]_{\mathrm{PGL}(3)}$ from a finite number of values of ϕ (a.k.a. images). This problem may be decomposed into two parts:

Profile Interpolation:
given a finite set of points of images,
compute the profile.

Objectivization:
given the profile of O,
compute $[O]_{\mathrm{PGL}(3)}$.

Photogrammetry Algorithm for 6 Points in the Plane

Profile Interpolation:

Given 4 images in $S_3 \subset \mathbb{P}^4$, interpolate hyperplane H.
[The profile is $S_3 \cap H$.]

Objectivization:

Let p_H the point in $(\mathbb{P}^4)^*$ dual to H.
[In general, $p_H \notin F_4$.]
Return $\gamma^{-1}(p_H) \in M_6^2$ (two solutions).

2D Pictures of 6 Points in 3-Space

The photographic map

$$\phi : (\mathbb{P}_3)^6 \times \mathbb{P}^2 \dashrightarrow M_6^2,$$

$$((p_1, \ldots, p_6), p_0) \mapsto [\pi_{p_0}(p_1, \ldots, p_6)]_{\mathrm{PGL}(3)}$$

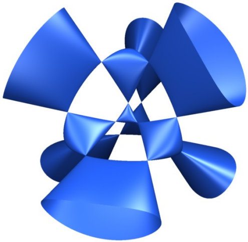
assigns to each object of 6 points in the plane and camera position in the plane a picture of 6 points in the plane, up to equivalence. The first 5 components are given as products of determinants:

$$x_1 = [0123][0456], \ldots, x_5 = [0135][0246].$$

These are quadratic forms vanishing in $p_1, \ldots, p_6$.
The sixth component is a quartic form with double points in $p_1, \ldots, p_6$.

Geometry of the Profile

While M_6^2 is a double cover of $\mathbb{P}^4$ branched over the Igusa quartic, the profile is a double cover branched of a hyperplane $\mathbb{P}^3$ branched over a Kummer surface K.



The Kummer surface has 16 double points, 15 are intersections with the singular lines of the Igusa quartic. The 16th is a point where the hyperplane is tangential to the Igusa quartic.

Photogrammetry Algorithm for 6 Points in 3-space

Profile Interpolation:

Given 3 images in M_6^2, apply γ to get 3 points in $\mathbb{P}^4$.
Let L be the intersection of the 3 dual hyperplanes (a line).
Let p be a point in the intersection $L \cap S_3$ (three solutions).
Let H_p be the dual hyperplane in $\mathbb{P}^4$.
[H_p is a hyperplane containing the 3 images and tangent to F_4.]
[The profile is $\gamma^{-1}(H_p)$.]

Objectivization:

Just return the point p dual to H_p.

Conclusion

Using profiles once can solve various problems in photogrammetry.

The method does not scale well with the size of the object, because the quotient varieties quickly become very complicated.

The method scales better with the number of images (interpolating with more points). Additional images can easily be used to balance the error.

Sometimes the profile does not determine the object (cf Kahl's talk).