Disturbance Decoupled Observer Design for Descriptor Systems.

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Abstract

In this paper we study the observer design problem for descriptor systems with partly unknown inputs. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupled estimation problem with or without stable error spectrum requiring at the same time that the resulting combined system is regular and of index at most one. All results are proved based on a condensed form that can be computed using orthogonal matrix transformations, i.e., transformations that can be implemented in a numerically stable way.

Keywords: Descriptor system, error spectrum, stability, disturbance decoupled estimation, index, orthogonal matrix transformation.

AMS subject classification: 93B05, 93B40, 93B52, 65F35

1 Introduction

In two recent papers [9, 10] the disturbance decoupling problem for descriptor systems has been studied via algebraic methods that make use of orthogonal matrix transformations that allow implementation as numerically stable algorithms. In this paper we follow this approach and study observer design with unknown inputs for linear descriptor systems of the form

\[
\begin{align*}
E \dot{x} &= Ax + Bu + Gq, \quad x(t_0) = x^0 \\
y &= Cx, \\
z &= Hx.
\end{align*}
\]

Here \( y, u \) are observations, \( z \) is an estimated output and \( x^0 \) a given initial value. The system matrices satisfy \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n}, H \in \mathbb{R}^{l \times n} \). The term \( q(t) \) represents a disturbance, which may represent modelling or measuring errors, noise, higher order terms in linearization or just an unknown input to the system. In this paper, we only study square systems \((E, A)\) are square\), for a reduction of the general case to the square case see [8].

Consider the construction of an observer for system (1) of the form

\[
\begin{align*}
E_\varepsilon \dot{w} &= A_\varepsilon w + Ky + Su, \quad w(t_0) = w^0 \\
\dot{z} &= Fw
\end{align*}
\]

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with \(E_c, A_c \in \mathbb{R}^{n_c \times n_c}, F \in \mathbb{R}^{k \times n_c}, K \in \mathbb{R}^{n_c \times q} \) and \(S \in \mathbb{R}^{n_c \times m}\). Combining (1) and (2) we obtain a resulting combined system

\[
\dot{E} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = A \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B & G \\ S & 0 \end{bmatrix} \begin{bmatrix} u \\ q \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ w(t_0) \end{bmatrix} = \begin{bmatrix} x^0 \\ w^0 \end{bmatrix}
\]

(3)

and an estimation error

\[
z - \hat{z} = \begin{bmatrix} H & -F \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.
\]

(4)

Here

\[
\begin{bmatrix} E \\ E_c \end{bmatrix}, \begin{bmatrix} A \\ 0 \end{bmatrix}, \begin{bmatrix} A_c \n C & A_c \end{bmatrix}.
\]

(5)

If \((\mathcal{E}, A)\) is a regular pencil, i.e., \(\det(s\mathcal{E} - A)\) does not vanish identically for all \(s \in \mathbb{C}\), then (3) has a unique solution for all sufficiently smooth inputs \(u, q\) and all consistent initial values \(x^0, w^0\), see [4]. (An initial value is called consistent if a continuously differentiable solution of the initial value problem exists.) The smoothness requirements depend on the index of the system, i.e., the size of the largest Jordan block to the eigenvalue infinity of \((\mathcal{E}, A)\). For an index \(k\) system we need the input function to be at least \(k - 1\) times differentiable if we want to avoid impulsive solutions, see [13, 4]. Since, in general, we cannot assume that \(q\) is differentiable and since usually the input functions are only piecewise continuous, it is desirable that the combined system (3) has index at most one. For a numerical method that implements an observer, this is even an essential requirement, since in the presence of roundoff errors and due to the fact that numerical methods work with discrete approximations, impulses in the solution in general cannot be handled adequately.

It should be pointed out, furthermore, that although an index one system can always be transformed to an equivalent standard system by solving a linear system, from a numerical point of view such a transformation is disastrous. First of all the solution of the algebraic equations may introduce large numerical errors, due to ill-conditioning and secondly, since the algebraic constraint is not there any more, numerical integration methods will slowly drift off the algebraic constraint, see [4]. So to obtain accurate results it is essential to work with the index one system rather than a mathematically equivalent reduced order standard system.

Since the system (3) is block triangular, the index-one requirement implies that both diagonal blocks, and in particular our original system has to be regular and of index at most one. Even though this is not the case in most applications like multi-body systems dynamics or circuit simulation, this property is easily achieved by deflating uncontrollable or unobservable impulsive modes and by computing an appropriate initial feedback that can be obtained in a numerically stable way, see the recent papers [5, 6, 7], and also [9, 10]. It is therefore not restrictive if we assume in the following that \((E, A)\) is regular and of index at most one.

The transfer function \(F(s)\) from \(\begin{bmatrix} u \\ q \end{bmatrix}\) to \(z - \hat{z}\) in (3), (4) is given by

\[
F(s) := \begin{bmatrix} H & -F \end{bmatrix} \begin{bmatrix} 0 & sE - A \\ -KC & sE_c - A_c \end{bmatrix}^{-1} \begin{bmatrix} B & G \\ S & 0 \end{bmatrix}.
\]

(6)

**Definition 1** We say that (2) is a disturbance decoupled observer of system (1) if the combined system (3) is uniquely solvable for all piecewise continuous input functions and all
consistent initial values $x^0, w^0$, and if furthermore the observation input $u$ and the disturbance $q$ have no influence on the estimation error $z - \hat{z}$. The integer $n_c$, i.e., the dimension of $E$, is called the order of the observer (2).

The disturbance decoupling problem and the disturbance decoupled observer problem for standard systems $E = I$ have been solved using an elegant geometric approach in [3, 18]. This geometric approach has been generalized to study the disturbance decoupling problem for descriptor systems in [14, 12, 2]. But, the results in [14, 12, 2] are very complicated and far from complete. In particular, the index and stability of the resulting combined system (3) and the numerical aspects of computing the desired observer have not been considered in these papers. Note that for the development of reliable numerical methods it is essential that the subspace computations are performed via orthogonal equivalence transformations, since this is essentially the only way to guarantee numerical stability of the method. The geometric solution methods, although very elegant and coordinate free, usually do not satisfy these requirements and hence are not suited for numerical computation. The need for reliable numerical methods was already pointed out for the standard case in [18].

It is the subject of this paper to present necessary and sufficient conditions for the existence of disturbance decoupled observers that lead to a system (3) which is regular and of index at most one. These necessary and sufficient condition are given in terms of the original data and numerically stable procedures to check these conditions are discussed.

Furthermore, we also derive necessary and sufficient conditions for which the system (3) has stable error spectrum, i.e., the estimation error $z - \hat{z}$ satisfies

$$\lim_{t \to \infty} (z(t) - \hat{z}(t)) = 0$$

for all consistent initial values $x^0, w^0$.

The basis for our results is a condensed form under orthogonal equivalence transformations, which we will describe in the next section. The computation of this form, which is a variation of the generalized upper triangular form for matrix pencils [11], can be directly implemented as a numerically stable algorithm. The main results are proved constructively and the numerical algorithm based on orthogonal transformations for computing the required observer is embedded in the proofs.

2 Preliminaries

We use the following notation, see also [5].

- $S_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix $M$;
- $T_\infty(M)$ denotes a matrix with orthogonal columns spanning the right nullspace of a matrix $M^T$;
- $M^-$ denotes the orthogonal complement of the space spanned by the columns of $M$;
- $\deg(f(s))$ denotes the degree of the polynomial $f(s)$;
- $\text{rank}_g[M](s)$ denotes the generic rank of a rational matrix function.
\begin{itemize}
\item $\mathbb{C}^+$ denotes the closed right half complex plane.
\item $\mathbb{C}^-$ denotes the open left half complex plane.
\item For convenience we do not distinguish between a matrix with orthogonal columns and the space spanned by its columns.
\end{itemize}

Using this notation we have the following characterization of a disturbance decoupled observer for system (1).

**Lemma 2** An observer of the form (2) is a disturbance decoupled observer for system (1) if and only if the matrix pencil \((E, A)\) defined in (5) is regular and

\[
\text{rank}_g \begin{bmatrix}
  sE - A & 0 & B & G \\
  -KC & sE_c - A_c & S & 0 \\
  H & -F & 0 & 0
\end{bmatrix} = n + n_c,
\]

where \(n_c\) is the order of observer (2).

**Proof.** An observer of the form (2) is a disturbance decoupled observer for system (1) if and only if the matrix pencil \((E, A)\) is regular and the transfer function \(F(s)\) as in (6) is zero. Since \((E, A)\) is regular if and only if \((E, A_c)\) and \((E, A)\) are regular, the proof follows directly from the fact that

\[
\text{rank}_g \begin{bmatrix}
  sE - A & 0 & B & G \\
  -KC & sE_c - A_c & S & 0 \\
  H & -F & 0 & 0
\end{bmatrix} = \text{rank}_g F(s) + (n + n_c).
\]

\[\square\]

In order to make the present paper self-contained, we recall the following result, e.g. [5], which characterizes the pencils that are regular and of index at most one.

**Lemma 3** Given \(E, A \in \mathbb{R}^{n \times n}\). The pencil \((E, A)\) is regular and of index at most one if and only if

\[
\text{deg}(\det(sE - A)) = \text{rank}(E). \tag{8}
\]

Moreover, if (8) holds, then there exist nonsingular matrices \(X, Y \in \mathbb{R}^{n \times n}\) such that

\[
X(sE - A)Y = \begin{bmatrix}
\text{rank}(E) & n - \text{rank}(E) \\
\text{rank}(E) & n - \text{rank}(E)
\end{bmatrix}
\begin{bmatrix}
sE_1 - A_1 & -I \\
-I & -I
\end{bmatrix}
\]

with \(E_1\) nonsingular.

Given an arbitrary matrix pencil \((E, A)\), it is well-known [15, 17] that there exist nonsingular matrices that transform the pencil \((E, A)\) to Kronecker canonical form, which characterizes all the invariants of the pencil under equivalence. It is, however, in general impossible to compute the Kronecker canonical form with a finite precision algorithm, since this is an ill conditioned problem. Instead one can obtain a condensed form under orthogonal equivalence transformations, which also displays most of the desired invariants. This form, the generalized upper triangular form is well studied [11, 15] and has been implemented in LAPACK [1].
Lemma 4 [11] Given a matrix pencil \((E, A)\), \(E, A \in \mathbb{R}^{k \times n}\) there exist orthogonal matrices \(P \in \mathbb{R}^{k \times k}\), \(Q \in \mathbb{R}^{n \times n}\) such that the pencil \((P^T EQ, P^T AQ)\) is in the following generalized upper triangular form:

\[
P^T (sE - A)Q = \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 \\
    0 & 0 & 0 & 0 \\
    l_1 & 0 & 0 & 0 \\
    n_2 & 0 & 0 & 0 \\
    n_3 & 0 & 0 & 0 \\
    n_4 & 0 & 0 & 0 \\
    n_5 & 0 & 0 & 0 \\
    n_6 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} \\
    0 & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\
    0 & 0 & sE_{33} - A_{33} & sE_{34} - A_{34} \\
    0 & 0 & 0 & sE_{44} - A_{44}
\end{bmatrix}, \quad (9)
\]

where \(sE_{11} - A_{11}\) and \(sE_{44} - A_{44}\) contain all left and right singular Kronecker blocks of \(sE - A\), \(E_{22}\) and \(A_{33}\) are nonsingular, \(E_{33}\) is singular and \(\det(sE_{33} - A_{33}) = \det(-A_{33}) \neq 0\) for all \(s \in \mathbb{C}\), i.e., \(sE_{22} - A_{22}\) and \(sE_{33} - A_{33}\) contain the regular finite and infinite structure of \(sE - A\), respectively.

The form (9) allows to determine left and right reducing subspaces of a matrix pencil. These subspaces are uniquely defined, although the orthogonal matrices \(P\) and \(Q\) in (9) are not unique.

Definition 5 [11] Given a matrix pencil \((E, A)\), \(E, A \in \mathbb{R}^{k \times n}\). Let \(P, Q\) be orthogonal matrices, such that \(P^T (sE - A)Q\) is of the form (9).

(i) The left and right reducing subspaces \(V_{l \rightarrow l}[E, A]\) and \(V_{l \rightarrow r}[E, A]\) of \((E, A)\) corresponding to the finite spectrum of \((E, A)\) are the spaces spanned by the leading \(l_1 + n_2\) columns of \(P\) and leading \(n_1 + n_2\) columns of \(Q\), respectively.

(ii) The maximal left and right reducing subspaces \(V_{m \rightarrow l}[E, A]\) and \(V_{m \rightarrow r}[E, A]\) are the spaces spanned by the leading \(l_1 + n_2 + n_3\) columns of \(P\) and leading \(n_1 + n_2 + n_3\) columns of \(Q\), respectively.

Lemma 4 and the numerical method to compute the form (9) can be employed to obtain the following condensed form for \((E, A, B, C, G, H)\).

Theorem 6 Given a system of the form (1) with \((E, A)\) regular and of index at most one, there exist orthogonal matrices \(U, V \in \mathbb{R}^{n \times n}\) such that

\[
U^T (sE - A)V
\]

\[
= \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    l_1 & 0 & 0 & 0 & 0 & 0 \\
    n_2 & 0 & 0 & 0 & 0 & 0 \\
    n_3 & 0 & 0 & 0 & 0 & 0 \\
    n_4 & 0 & 0 & 0 & 0 & 0 \\
    n_5 & 0 & 0 & 0 & 0 & 0 \\
    n_6 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} & sE_{15} - A_{15} & sE_{16} - A_{16} \\
    0 & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} & sE_{25} - A_{25} & sE_{26} - A_{26} \\
    0 & 0 & sE_{33} - A_{33} & sE_{34} - A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} \\
    0 & 0 & 0 & sE_{44} - A_{44} & 0 & 0
\end{bmatrix}
\]

\[
CV = \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
    0 & 0 & 0 & C_4 & C_5 & C_6
\end{bmatrix}, \quad HV = \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
    0 & 0 & H_3 & H_4 & H_5 & H_6
\end{bmatrix},
\]

\[
U^T G = \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad U^T B = \begin{bmatrix}
    n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\
    G_1 & B_1 & B_2 & B_3 & B_4 & B_5
\end{bmatrix}, \quad (10)
\]
where $E_{35}$ and $A_{46}$ are nonsingular, $H_3$ and $C_4$ are of full column rank and, furthermore,

\[
\begin{align*}
\text{rank} \begin{bmatrix} E_{11} & G_1 \end{bmatrix} &= i_1, \\
\text{rank} \begin{bmatrix} sE_{22} - A_{22} \end{bmatrix} &= n_2, \quad \forall s \in \mathbb{C}, \\
\text{rank} \begin{bmatrix} -A_{33} & -A_{34} & sE_{35} - A_{35} \\ 0 & C_4 & C_5 \\ H_3 & H_4 & H_5 \end{bmatrix} &= n_3 + n_4 + n_5, \quad \forall s \in \mathbb{C}, \\
\text{rank} \begin{bmatrix} 0 & C_4 & C_5 & C_6 \\ H_3 & H_4 & H_5 & H_6 \end{bmatrix} &= n_3 + n_4 + n_5 + n_6.
\end{align*}
\]

**Proof.** Performing a row compression of $G$ and a column compression of $\begin{bmatrix} C \\ H \end{bmatrix}$ first and then applying Lemma 4 to $T^T_\infty(G)(sI - A)S_\infty(\begin{bmatrix} C \\ H \end{bmatrix})$, we can determine orthogonal matrices $U_1$ and $V_1$, such that

\[
U^T_1 (sE - A)V_1 = \begin{bmatrix} i_1 & n_2 & \tilde{n}_3 \\ n_1 & 0 & 0 \\ \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 \end{bmatrix}
\]

\[
U^T_1 G = \begin{bmatrix} i_1 & 0 & 0 \\ n_1 & 0 & 0 \\ \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 \end{bmatrix}, \quad CV_1 = \begin{bmatrix} 0 & 0 & \tilde{C}_3 \\ \tilde{n}_3 & \tilde{n}_3 \end{bmatrix},
\]

\[
HV_1 = \begin{bmatrix} \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 \\ 0 & 0 & \tilde{H}_3 \end{bmatrix},
\]

where

\[
\text{rank} \begin{bmatrix} E_{11} & G_1 \end{bmatrix} = i_1, \quad \text{rank} \begin{bmatrix} \tilde{E}_{33} \\ \tilde{C}_3 \\ H_3 \end{bmatrix} = \tilde{n}_3,
\]

and

\[
\text{rank} \begin{bmatrix} sE_{22} - A_{22} \end{bmatrix} = n_2, \quad \text{rank} \begin{bmatrix} s\tilde{E}_{33} - \tilde{A}_{33} \\ \tilde{C}_3 \\ H_3 \end{bmatrix} = \tilde{n}_3, \quad \forall s \in \mathbb{C}.
\]

Using Lemma 4 we can determine the generalized upper triangular form of $s\tilde{E}_{33} - \tilde{A}_{33}$, i.e., there exist orthogonal matrices $U_2$ and $V_2$ such that

\[
\begin{bmatrix} I_{i_1+n_2} \\ U_2^T \end{bmatrix} \begin{bmatrix} s\tilde{E}_{13} - \tilde{A}_{13} \\ s\tilde{E}_{23} - \tilde{A}_{23} \\ s\tilde{E}_{33} - \tilde{A}_{33} \end{bmatrix} V_2 = \begin{bmatrix} i_1 & 0 & 0 \\ n_2 & \tilde{n}_3 & 0 \\ n_5 & \tilde{n}_3 & \tilde{n}_5 \\ \tilde{n}_6 & \tilde{n}_5 & \tilde{n}_6 \end{bmatrix},
\]

\[
\begin{bmatrix} s\Theta_{13} - \Phi_{13} & sE_{16} - A_{16} \\ s\Theta_{43} - \Phi_{43} & sE_{26} - A_{26} \\ s\Theta_{33} - \Phi_{33} & sE_{36} - A_{36} \\ 0 & sE_{46} - A_{46} \end{bmatrix},
\]
with $\Theta_{33}$ of full row rank and $sE_{46} - A_{46}$ of full column rank for all $s \in \mathbb{C}$. Set

$$
\hat{C}_3 V_2 := \begin{bmatrix} \hat{n}_3 - n_6 & n_6 \\ \Psi_3 & C_6 \end{bmatrix}, \quad \hat{H}_3 V_2 := \begin{bmatrix} \hat{n}_3 - n_6 & n_6 \\ \Pi_3 & H_6 \end{bmatrix}.
$$

By (15) we have that

$$
\begin{bmatrix}
\Theta_{33} & E_{36} \\
0 & E_{46} \\
\Psi_3 & C_6 \\
\Pi_3 & H_6
\end{bmatrix}
$$

is of full column rank. Hence, there exists an orthogonal matrix $V_3$ such that

$$
\begin{bmatrix}
s\Theta_{13} - \Phi_{13} \\
s\Theta_{23} - \Phi_{23} \\
s\Theta_{33} - \Phi_{33}
\end{bmatrix}
\begin{bmatrix}
V_3 \\
\Psi_3 V_3 := \begin{bmatrix} 0 & C_4 & C_5 \end{bmatrix}, \quad \Pi_3 V_3 := \begin{bmatrix} H_3 & H_4 & H_5 \end{bmatrix}
\end{bmatrix} = \begin{bmatrix} n_3 & n_4 & n_5 \\
n_4 & n_4 & n_5 \\
n_5 & n_4 & n_5
\end{bmatrix}
$$

with $E_{35}$ nonsingular and $H_3, C_4$ of full column rank. Let

$$
U = U_1 \begin{bmatrix} I_{l_1} \\
U_2
\end{bmatrix}, \quad V = V_1 \begin{bmatrix} I_{n_1} \\
V_2
\end{bmatrix} \begin{bmatrix} I_{n_5} \\
V_3 \\
I_{n_5}
\end{bmatrix}.
$$

In order to prove that $U$ and $V$ give the transformation matrices to the condensed form (10), we only need to prove that $l_4 = n_6$, $E_{46} = 0$ and that $A_{46}$ is nonsingular.

Since $(E, A)$ is regular we have that rank$_2(sE_{46} - A_{46}) = l_4$, but rank$(sE_{46} - A_{46}) = n_6$ for all $s \in \mathbb{C}$. Hence, we have $l_4 = n_6$, $A_{46}$ is nonsingular and det$(sE_{46} - A_{46}) = \det(-A_{46}) \neq 0$. Furthermore, $(E, A)$ is of index at most one. By Lemma 3, this implies that rank$(E) = \text{deg}(\det(sE - A))$, which gives that $E_{46} = 0$. $\Box$

Using the condensed form (10) we can characterize and compute the following subspaces.

Denote by $\Pi := T_\infty(G)$ the left nullspace of $G$, and by $\Gamma := S_\infty\left(\begin{bmatrix} C \\ H \end{bmatrix}\right)$ the right nullspace of the combined output matrices $\begin{bmatrix} C \\ H \end{bmatrix}$. By Definition 5 we can characterize the orthogonal complements of the maximal reducing subspaces of the projected pencil $(\Pi^T E \Gamma, \Pi^T A \Gamma)$, i.e.,

$$
\Phi_1 := V^{-\text{red}}_{m+}[\Pi^T E \Gamma, \Pi^T A \Gamma], \quad \Phi_2 := V^{-\text{red}}_{m+}[\Pi^T E \Gamma, \Pi^T A \Gamma].
$$

Note that the projected pencil $(\Pi^T E \Gamma, \Pi^T A \Gamma)$ corresponds to the part of the system that is not influenced by the disturbance and which cannot be observed or estimated in the output. It is clear that the properties of this part have to be feasible with the required system properties in the combined system (3) in order to obtain the desired observer.

An equally important role will be played by the projected pencil

$$
s\hat{E} - \hat{A} := s\Phi_1^T \Pi^T E - \Phi_1^T \Pi^T A,
$$

(17)
in particular the matrix \( \hat{E} \). As we will show below, with \( \mu := \text{rank}(\hat{E}) \), we have that \( \mu + l \) is the order of the observer.

We will also need the quantities

\[
\begin{align*}
\xi & := \text{rank}\begin{bmatrix} C \\ H \end{bmatrix} + \text{dim}(\Phi_r), \\
\tau & := \text{rank}\begin{bmatrix} \hat{E} \\ C \\ H \end{bmatrix} V_{f-r}[\hat{E}, \hat{A}]) - \text{rank}\begin{bmatrix} \hat{E} \\ C \end{bmatrix} V_{f-r}[\hat{E}, \hat{A}] - \text{rank}(\hat{E}).
\end{align*}
\] (18)

Our main theorem below shows that we need \( \tau = 0 \) to have a solution of the observer design problem. In a similar way the quantity \( \xi \) describes the dimension of the subproblem for which a stabilizability condition has to be required, see Theorem 8.

The following corollary shows that these important quantities can be directly read off from the condensed form (10), i.e., they can be determined via a numerically stable procedure.

**Corollary 7** Let \( E, A, C, G, H \) and \( B \) be in the condensed form (10). Then

\[
\begin{align*}
\xi &= n_3 + n_4 + n_5 + n_6, \\
\mu &= n_5, \\
\tau &= n_3.
\end{align*}
\] (19)

In the next section we derive necessary and sufficient conditions for the existence of a disturbance decoupled observer.

### 3 Main Theorem

In this section we now present our main Theorem.

**Theorem 8** Given a system of the form (1) with \((E, A)\) regular and of index at most one. Let the matrices \( \hat{E}, \hat{A} \) be defined as in (17) and the indices \( \xi, \tau, \mu \) as in (18).

(i) System (1) has a disturbance decoupled observer of the form (2) with a regular, index at most one pencil \((\hat{E}, \hat{A})\) as in (5) if and only if \( \tau = 0 \).

(ii) System (1) has a disturbance decoupled observer of the form (2) with a regular, index at most one pencil \((\hat{E}, \hat{A})\) as in (5) and has stable error spectrum if and only if \( \tau = 0 \) and

\[
\text{rank}\begin{bmatrix} s\hat{E} - \hat{A} \\ C \end{bmatrix} = \xi, \quad \forall s \in \mathbb{C}^+.
\] (20)

Moreover, the order of the observer in both cases can be chosen to be \( \mu + l \).

**Proof.** The proof is quite technical but has the advantage that for the sufficiency part we explicitly construct the desired observers and hence the proof immediately gives a computational method for the observer design.
We may assume without loss of generality that the system is in the condensed form (10). Then by Corollary 7 we have that \( \tau = 0 \) if and only if \( n_3 = 0 \) and condition (20) translates to
\[
\begin{bmatrix}
-A_{33} & -A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} \\
0 & 0 & 0 & -A_{46} \\
0 & C_4 & C_5 & C_6 
\end{bmatrix}
= n_3 + n_4 + n_5 + n_6, \quad \forall s \in \mathbb{C}^+.
\] (21)

We first prove the necessity in (i) and (ii) and then we prove sufficiency for both cases by explicitly constructing the observer.

(i) **Necessity:** Assume that system (1) has a disturbance decoupled observer of the form (2) with a regular matrix pencil \((\mathcal{E}, \mathbf{A})\) of index at most one. We have to show that \( n_3 = 0 \).

Both matrix pencils \((E, A)\) and \((E, A_e)\) are regular and of index at most one. Applying Lemma 3, we may assume without loss of generality that
\[
sE_e - A_e = \begin{bmatrix} n_{e1} & n_{e2} \\ n_{e1} & n_{e2} \end{bmatrix} \begin{bmatrix} sE_{e1} - A_{e1} & -I \\ -I & -I \end{bmatrix},
\] (22)
with \( E_{e1} \) nonsingular and that the matrices \( S, K, F \) are partitioned accordingly as
\[
S = \begin{bmatrix} n_{e1} & n_{e2} \\ n_{e1} & n_{e2} \end{bmatrix}, \quad K = \begin{bmatrix} n_{e1} & n_{e2} \\ n_{e1} & n_{e2} \end{bmatrix}, \quad F = \begin{bmatrix} n_{e1} & n_{e2} \\ n_{e1} & n_{e2} \end{bmatrix}.
\]

Using Schur complements with the nonsingular matrices \( E_{35} \) and \( A_{46} \) we obtain the submatrix \( M(s) \) given by
\[
\begin{bmatrix}
-A_{33} & -A_{34} & sE_{35} - A_{35} & 0 & B_3 + (A_{35} E_{35}^{-1} E_{36} - A_{36}) A_{46}^{-1} B_4 \\
0 & -K_1 C_4 & -K_1 C_5 & sE_{e1} - A_{e1} & S_1 + (K_1 C_5 E_{35}^{-1} E_{36} - K_1 C_6) A_{46}^{-1} B_4 \\
H_3 & H_4 + F_2 K_2 C_4 & H_5 + F_2 K_2 C_5 & -F_3 & \hat{H}_6 
\end{bmatrix}
\]
where
\[
\hat{H}_6 := \{(H_6 - H_4 E_{35}^{-1} E_{36}) + F_2 K_2 (C_6 - C_5 E_{35}^{-1} E_{36}) A_{46}^{-1} B_4 - F_2 S_1 \}.
\]

By (7), we have
\[
n + n_{e1} + n_{e2} = l_1 + n_2 + n_{e2} + n_6 + \text{rank}_g(M(s)).
\]
But we know that \( n = l_1 + n_2 + n_5 + n_6 \). Thus, \( \text{rank}_g(M(s)) = n_5 + n_{e1} \) and hence, we have that
\[
\begin{bmatrix}
H_3 & H_4 + F_2 K_2 C_4 & \hat{H}_6 
\end{bmatrix}
= 0.
\]
But \( H_3 \) is of full column rank, so we necessarily have \( n_3 = 0 \), i.e., \( \tau = 0 \).

(ii) **Necessity:** By (i) we have already \( n_3 = 0 \). Since (7) holds and \((\mathcal{E}, \mathbf{A})\) is regular and of index at most one, we have that
\[
n + n_e \leq \text{rank}_g \begin{bmatrix}
sE - A & 0 & G \\
-KC & sE_e - A_e & 0 \\
H & -F & 0 
\end{bmatrix}
\] \leq \text{rank}_g \begin{bmatrix}
sE - A & 0 & B & G \\
-KC & sE_e - A_e & S & 0 \\
H & -F & 0 & 0 
\end{bmatrix}
= n + n_e.
\]
Hence, we obtain in the condensed form (10) that
\[
\text{rank}_2 \begin{bmatrix}
-A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\
0 & 0 & -A_{46} & 0 \\
-KC_4 & -KC_5 & -KC_6 & sE_e - A_e \\
H_4 & H_5 & H_6 & -F
\end{bmatrix} = n_3 + n_6 + n_e.
\]  

(23)

Computing a column compression \([H_4 \ H_5 \ H_6 \ -F]\) followed by the computation of the generalized upper triangular form of
\[
\begin{bmatrix}
-A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\
0 & 0 & -A_{46} & 0 \\
-KC_4 & -KC_5 & -KC_6 & sE_e - A_e
\end{bmatrix}
S_\infty \left( \begin{bmatrix} H_4 & H_5 & H_6 & -F \end{bmatrix} \right),
\]
we can determine orthogonal matrices \(P\) and \(Q\) such that
\[
PT \begin{bmatrix}
-A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\
0 & 0 & -A_{46} & 0 \\
-KC_4 & -KC_5 & -KC_6 & sE_e - A_e
\end{bmatrix} Q
= \frac{\tilde{t}_1}{\tilde{t}_2} \begin{bmatrix}
-s\Theta_{34} - \Phi_{34} & s\Theta_{35} - \Phi_{35} & s\Theta_{36} - \Phi_{36} \\
0 & s\Theta_{45} - \Phi_{45} & s\Theta_{46} - \Phi_{46}
\end{bmatrix},
\]
\[
\begin{bmatrix} H_4 & H_5 & H_6 & -F \end{bmatrix} Q = \begin{bmatrix} 0 & 0 & \Psi_6
\end{bmatrix}
\]
where \(\Theta_{34}\) has full row rank, \(\Psi_6\) has full column rank and \(s\Theta_{45} - \Phi_{45}\) has full column rank for all \(s \in C\). Then using (23) and (24) we obtain
\[
\hat{t}_1 + \hat{t}_2 = n_3 + n_6 + n_e = \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3,
\]
i.e., we have \(\tilde{t}_2 = t_2 + t_3\) and hence, the matrix \(\begin{bmatrix} s\Theta_{45} - \Phi_{45} & s\Theta_{46} - \Phi_{46} \end{bmatrix}\) is square. Furthermore, it is easy to see that
\[
\begin{bmatrix}
-s\Theta_{45} - \Phi_{45} & s\Theta_{46} - \Phi_{46} \\
0 & \Psi_6
\end{bmatrix}
\]
has full column rank for all \(s \in C\). Thus if the error spectrum of the disturbance decoupled observer (2) is stable, then
\[
\begin{bmatrix} \Theta_{45} & \Theta_{46} \\
\Phi_{45} & \Phi_{46}
\end{bmatrix}
\]
is stable, i.e.,
\[
\text{rank} \begin{bmatrix} s\Theta_{45} - \Phi_{45} & s\Theta_{46} - \Phi_{46} \end{bmatrix} = t_2 + t_3, \quad \forall s \in \bar{C}^+.
\]
We have
\[
\begin{bmatrix}
-A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\
0 & 0 & -A_{46} & 0 \\
-KC_4 & -KC_5 & -KC_6 & sE_c - A_c \\
H_A & H_5 & H_6 & -F \\
C_4 & C_5 & C_6 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
= n_6 + n_e + \text{rank} \begin{bmatrix}
-A_{34} & sE_{35} - A_{35} \\
H_A & H_5 \\
C_4 & C_5
\end{bmatrix}
\]
\[
= n_4 + n_5 + n_6 + n_e, \quad \forall s \in \mathbb{C}.
\]

Therefore, if we set
\[
\begin{bmatrix}
C_4 & C_5 & C_6 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} Q = \begin{bmatrix} t_1 & t_2 & t_3 \\
\Pi_4 & \Pi_5 & \Pi_6
\end{bmatrix},
\]
then we have
\[
\text{rank} \begin{bmatrix}
s\Theta_{34} - \Phi_{34} \\
\Pi_4
\end{bmatrix} = t_1, \quad \forall s \in \mathbb{C}.
\]

Furthermore, we obtain
\[
\begin{bmatrix}
-A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 \\
0 & 0 & -A_{46} & 0 \\
-KC_4 & -KC_5 & -KC_6 & sE_c - A_c \\
C_4 & C_5 & C_6 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
= \text{rank} \begin{bmatrix}
s\Theta_{34} - \Phi_{34} & s\Theta_{35} - \Phi_{35} & s\Theta_{36} - \Phi_{36} \\
0 & s\Theta_{45} - \Phi_{45} & s\Theta_{46} - \Phi_{46} \\
\Pi_4 & \Pi_5 & \Pi_6
\end{bmatrix}
\]
\[
= t_1 + t_2 + t_3
\]
\[
= n_4 + n_5 + n_6 + n_e, \quad \forall s \in \tilde{\mathbb{C}}^+.
\]

Thus Corollary 7 implies that \( \text{rank} \begin{bmatrix} s\tilde{E} - \tilde{A} \\
C
\end{bmatrix} = \xi \) for all \( s \in \tilde{\mathbb{C}}^+ \).

(i) and (ii) **Sufficiency:** Let \( E_c, A_c, K, S \) and \( F \) as in (2) be of the form (22) with
\[
K_1C_4 = -A_{34}, \quad K_2C_4 = -H_4, \\
E_{c1} = E_{35}, \quad A_{c1} = A_{35} + K_1C_5, \\
F_1 = -(H_5 + K_2C_5), \quad F_2 = I \in \mathbb{R}^{i \times i}, \\
S_1 = \{(A_{36} - A_{35}E_{35}^{-1}E_{36}) + K_1(C_6 - C_5E_{35}^{-1}E_{36})\}A_{46}^{-1}B_4 - B_3, \\
S_2 = \{(H_6 - H_5E_{35}^{-1}E_{36}) + K_2(C_6 - C_5E_{35}^{-1}E_{36})\}A_{46}^{-1}B_4.
\]

Note that \( C_4 \) is of full column rank, so there exist solutions \( K_1, K_2 \) in (25). Since \((E, A)\) is regular and of index at most one, \( E_{35} \) is nonsingular, hence, a simple calculation yields that
the matrix pencil \( (E, A) \) is regular and of index at most one. Furthermore, from \( \tau = n_3 = 0 \), we have

\[
\begin{bmatrix}
  sE - A & 0 & B & G \\
  -KC & sE_c - A_c & S & 0 \\
  H & -F & 0 & 0
\end{bmatrix}
\]

\[
\text{rank } \begin{bmatrix}
  -A_{34} & sE_{35} - A_{35} & sE_{36} - A_{36} & 0 & 0 & B_3 \\
  0 & -A_{45} & 0 & 0 & B_4 \\
  A_{34} & -K_1C_5 & -K_1C_6 & sE_{35} - A_{35} - K_1C_5 & 0 & S_1 \\
  H_4 & -K_2C_5 & -K_2C_6 & 0 & -I & S_2 \\
  H_4 & H_5 & H_6 & H_5 + K_2C_5 & -I & 0
\end{bmatrix}
\]

\[
= l_1 + n_2 + \text{rank } \begin{bmatrix}
  sE_{35} - A_{35} - K_1C_5 & (A_{35} + K_1C_5)E_{35}^{-1}E_{36} - A_{36} - K_1C_6 & B_3 + S_1 \\
  0 & -A_{45} & 0 & B_4 \\
  H_5 + K_2C_5 & H_6 + K_2C_6 & (H_5 + K_2C_5)E_{35}^{-1}E_{36} & -S_2
\end{bmatrix}
\]

\[
= l_1 + n_2 + (n_5 + l) + (n_5 + n_6) = n + (n_5 + l),
\]

i.e., (7) holds. Therefore, the proof of sufficiency in part (i) is complete. Moreover, if \( \text{rank } \begin{bmatrix} sE - \hat{A} \\ C \end{bmatrix} = \xi \) for all \( s \in \tilde{C}^+ \), then by (21) and a stabilizability theorem for descriptor systems [16], \( K_1 \) in (25) can be chosen such that \( (E_{35}, A_{35} + K_1C_5) \) is stable. For this \( K_1 \), (2) is a disturbance decoupled observer of system (1) with order \( n_5 + l \) and \( (E, A) \) is regular and of index at most one and also its error spectrum, which is a subset of the finite spectrum of

\[
\begin{bmatrix}
  E_{35} & E_{36} & 0 \\
  0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
  A_{35} + K_1C_5 & A_{36} + K_1C_6 & 0 \\
  0 & A_{45} & 0 \\
  K_2C_5 & K_2C_6 & I
\end{bmatrix},
\]

is stable. \( \square \)

**Remark 1** Theorem 8 shows that for descriptor systems of the form (1) with \( (E, A) \) regular and index of at most one, as for standard systems, the control input \( u \) (hence the matrix \( B \)) has no influence on the existence of the disturbance decoupled observers of the form (2) with \( (E_c, A_c) \) regular and index at most one.

**Remark 2** Note that the construction in the sufficiency part of the proof presents a design procedure for the desired observer that can be directly implemented as a numerical method. To design the observer, we need to solve in part i) two undetermined linear systems, which can be done in a numerically stable way via the singular value decomposition, [1]. In part ii) we have to solve a linear system for \( K_2 \) and a classical stabilization problem for \( K_1 \) for which also numerically stable methods are well-known, [16].

### 4 Conclusions

We have presented necessary and sufficient conditions for the existence of a disturbance decoupled observer for a descriptor system as in (1). All results are based on a condensed form which can be computed in a numerically stable way using orthogonal matrix transformations. Note that a result similar to Theorem 8 can be obtained in the same way if it is assumed that \( E_c \) is nonsingular.
References


