Dampening Controllers via a Riccati Equation Approach*

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Abstract

An algorithm is presented which computes a state feedback for a standard linear system which not only stabilizes, but also dampens the closed-loop system dynamics. In other words, a feedback gain matrix is computed such that the eigenvalues of the closed-loop state matrix are within the region of the left half-plane where the magnitude of the real part of each eigenvalue is greater than that of the imaginary part. This may be accomplished by solving a damped algebraic Riccati equation and a degenerate Riccati equation. The solution to these equations are computed using numerically robust algorithms.

Finally, the formulation of damped Riccati equations is unusual in that it may be viewed as an invariant subspace problem for a periodic Hamiltonian system. This periodic Hamiltonian system induces two damped Riccati equations, one which is associated with a symmetric solution and the other with a skew symmetric solution. These two solutions result in two different state feedbacks, both of which dampen the system dynamics, but produce different closed-loop eigenvalues, thus giving the controller designer greater freedom in choosing a desired feedback.

Keywords: linear quadratic controller, dampening feedback, damped dynamics, periodic systems, Riccati equation.

1 Introduction

Obtaining a stabilizing controller for a standard linear time-invariant system is a rather straight-forward problem; all that is necessary is to compute the stabilizing solution of an algebraic Riccati equation, or in a simplified case, a Lyapunov equation. In practice, however, it is often desirable to have a controller which ensures that the closed-loop dynamics are damped. Increased relative weighting of the input versus the state (or output) in the quadratic cost function of the Linear Quadratic Regulator problem often has little effect on the damping factor of the optimal feedback; it tends to move the poles of the closed-loop system further away from both the real and the imaginary axis. Another method, devised by Anderson and Moore [1], introduces a shift into the algebraic Riccati equation. The effect of this is to move the poles of the closed-loop system away from the imaginary axis, but does not necessarily guarantee that the closed-loop dynamics are damped. Other methods have recently been available to address directly the damping problem with a minimum degree of stability; however, these tend to be too complex algorithmically. For example, the algorithm presented by Kawasaki et al. [12] for placing poles in a left-open region bounded by a hyperbola requires an amount of floating point operations that is proportional to the fourth order of the dimension of the state vector. Methods based on Linear Matrix Inequalities [4] have even higher algorithmic
complexity. The end result is that often the control engineer is forced to place the poles of the closed-loop system to achieve the required damping. Unfortunately, pole-placement is usually an inherently ill-conditioned problem [7], and becomes impractical for large-order systems [8].

To circumvent these difficulties, we derive a new method which stabilizes a linear system such that the dynamics of the closed-loop system are damped, i.e., that the real part of each of the eigenvalues of the closed-loop system matrix is greater in magnitude than the imaginary part. This is accomplished, in part, by computing the a stable invariant subspace of a periodic Hamiltonian system associated with a particular damped algebraic Riccati equation. From this invariant subspace a Riccati solution is formed which moves the poles from outside to inside the damped region of the complex plane. This is based on the observation that a matrix with damped eigenvalues has anti-stable eigenvalues when squared. If a periodic system is used to describe the negative square of the closed-loop system, then the proposed damped Riccati equation can be used to stabilize this system.

The new method has a number of interesting properties. First, the algorithm produces two different Riccati solutions: one symmetric and one skew symmetric. While both of these solutions produce a dampening feedback, they have different properties. Second, the new method may be used in conjunction with standard stabilization methods via the solution of Lyapunov or Riccati equations, such as those mentioned in [8], and Anderson and Moore’s shifting method [1] to restrict the poles of the closed-loop system to a more complex region in the left half-plane. Third, all of the feedbacks mentioned may be computed using Schur methods to compute invariant subspaces [5, 9, 15]. For these methods numerical robustness has been demonstrated. They are also computationally efficient, requiring an amount of floating point operations that is proportional to the third order of the dimension of the state vector. Fourth, by varying the periodicity of the aforementioned periodic Hamiltonian system, it is possible to restrict the poles of the closed-loop system to ever narrower cones in the left half-plane. Finally, the dampening controller may be viewed as a controller that results from a particular choice of the state weighting matrix in the quadratic cost function of the standard linear quadratic regulator problem. This state weighting matrix provides valuable information about the states that need to be weighted more heavily in order to produce a dampening controller.
2 Damped Riccati Equations

Throughout this paper, we will be concerned with the computation of a feedback $u(t)$ which stabilizes the standard linear time–invariant system

$$
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
$$

(1)

Here $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, and we assume that the matrix pair $(A, B)$ is reachable, (i.e. $\text{Rank}[\lambda I - A, B] = n$ for all complex $\lambda$). Furthermore, we require the feedback to be proportional to the state $x$, i.e.,

$$
u(t) = Fx(t).
$$

(2)

Stabilizing feedbacks of this form may be straightforwardly obtained via a solution of an algebraic Riccati equation [11, 14]. Given any matrix $C \in \mathbb{R}^{p,n}$ such that the matrix pair $(A, C)$ is observable, (i.e., $(A^T, C^T)$ is reachable), then a stabilizing feedback in (2) is obtained as

$$
u(t) = -B^TWx(t),
$$

(3)

where $W$ is the symmetric positive semi-definite solution of the algebraic Riccati equation (ARE)

$$
0 = A^TW + WA - WBB^TW + C^TC,
$$

(4)

(see e.g. [15, 16]). An integral part of methods that compute an (optimal) stabilizing controller is the computation of a basis for an appropriate invariant subspace of a related Hamiltonian system, see e.g. [15, 18]. If the columns of the $2n \times n$ matrix

$$
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
$$

(5)

span the invariant subspace corresponding to the stable eigenvalues of the $2n \times 2n$ Hamiltonian matrix

$$
H = \begin{bmatrix}
A & -BB^T \\
-C^TC & -A^T
\end{bmatrix},
$$

(6)

and if $P_1$ is invertible, then the symmetric positive semi-definite solution $W$ of the algebraic Riccati equation in (4) is given by $W = P_2P_1^{-1}$, (see e.g. [15, 18]).

The computation of a dampening feedback can be carried out along the same lines. First, we introduce a linear zero–sum non-cooperative dynamic game from which arise two Riccati equations, whose solutions provide the required dampening feedback. The resulting minimax problem can be shown to be equivalent to a stable
invariant subspace identification problem for a periodic Hamiltonian system. The periodicity is essential in that it indirectly produces a stability region that is not the left half-plane, but rather a pair of cones in the complex plane, as shown by Figure 1.

![Damped Stability Region](image)

**Figure 1: Damped Stability Region**

Proceeding, we examine the following linear dynamic game. Consider linear systems of the form

\[
\begin{align*}
\dot{z} & = -(A^2 - BB^T C^T C) z + (A - I) B u + (A + I) B v, \quad z(0) = z_0 \\
y & = C (A - I) z \\
w & = C (A + I) z \\
\end{align*}
\]  

with quadratic cost functional

\[
\min_u \max_v \frac{1}{2} \int_0^\infty \left[ y^T y - w^T w + u^T u - v^T v \right] dt,
\]

where the matrices \( A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,m}, C \in \mathbb{R}^{p,n} \). The open-loop Nash equilibrium may be computed via the Hamilton–Jacobi–Bellman (or Issac’s) equation [2], which gives rise to a two-point boundary problem and a linear differential equation in the state \( z \) and costate \( q \)

\[
\begin{bmatrix}
\dot{z} \\
\dot{q}
\end{bmatrix} = H_X \begin{bmatrix}
z \\
q
\end{bmatrix} := \begin{bmatrix}
-(A^2 - RS) & AR + RA^T \\
SA + A^T S & (A^2 - RS)^T 
\end{bmatrix} \begin{bmatrix}
z \\
q
\end{bmatrix} 
\]

with

\[
\begin{align*}
z(0) = z_0, & \quad u = -B^T (A^T - I) q, & \quad R = BB^T, \\
q(\infty) = 0, & \quad v = B^T (A^T + I) q, & \quad S = C^T C. 
\end{align*}
\]

The matrix \( H_X \) can be factored as \( H_X = H_2 H_1 \) with

\[
H_2 := \begin{bmatrix}
-A & R \\
S & A^T
\end{bmatrix}, \quad H_1 := \begin{bmatrix}
A & -R \\
S & A^T
\end{bmatrix}.
\]
Since the matrix $H_X$ is Hamiltonian, the following Riccati equation is associated with $H_X$:

$$X(A^2 - RS) + (A^{2T} - SR)X - X(AR + RA^T)X + (A^T S + SA) = 0. \quad (12)$$

We call (12) the **Symmetric Damped Algebraic Riccati Equation (SDARE)**.

By changing the order of the factors, we obtain a related Hamiltonian

$$H_Y := H_1 H_2 = \begin{bmatrix} -(A^2 + RS) & AR - RA^T \\ SA - A^T S & (A^2 + RS)^T \end{bmatrix}. \quad (13)$$

Again there is an associated Riccati equation, called the **Skew-Symmetric Damped Algebraic Riccati Equation (SSDARE)**, which is given by

$$Y(A^2 + RS) + (A^{2T} + SR)Y - Y(AR - RA^T)Y + (A^T S - SA) = 0. \quad (14)$$

The solutions of these two Riccati equations are intimately connected to the stabilizing solution of the standard algebraic Riccati equation, as we demonstrate in the following theorem which is essentially the Theorem 5.1 in [19].

**Theorem 1**

a) If $H_X$ has no purely imaginary eigenvalues, then there exists an orthogonal matrix of the form $P = \begin{bmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{bmatrix}$, such that

$$P^T H_X P = \begin{bmatrix} T & G \\ 0 & -T^T \end{bmatrix},$$

where $T$ is upper quasi-triangular with eigenvalues in the open left half–plane and where $G = G^T$. Moreover if $P_1$ is nonsingular, then there exists a unique real symmetric matrix solution $X = -P_2 P_1^{-1}$ of (12) such that the closed–loop matrix $-A^2 + RS + (AR + RA^T)X$ is stable, i.e., all of its eigenvalues are in the open left half–plane.

b) If $H_Y$ has no purely imaginary eigenvalues then there exists an orthogonal matrix of the form $\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ -\tilde{P}_2 & \tilde{P}_1 \end{bmatrix}$, such that

$$\tilde{P}^T H_Y \tilde{P} = \begin{bmatrix} \tilde{T} & \tilde{G} \\ 0 & -\tilde{T}^T \end{bmatrix},$$

where $\tilde{T}$ is upper quasi-triangular with eigenvalues in the open left half–plane and where $\tilde{G} = \tilde{G}^T$. Moreover if $\tilde{P}_1$ is nonsingular, then there exists a unique real skew–symmetric matrix solution $Y = -\tilde{P}_2 \tilde{P}_1^{-1}$ of (14) such that the closed–loop matrix $-(A^2 + RS) + (AR - RA^T)Y$ is stable.
Proof. The real Hamiltonian-Schur decomposition follows directly from Theorem 5.1 in [19]. If \( P_1 \) is nonsingular, then \( X = -P_2P_1^{-1} \) is symmetric, since \( P \) is orthogonal implies that \( P_2P_1^T - P_1P_2^T = 0 \). Since \(-A^2 + RS + (AR + RA^T)X = P_1TP_1^{-1} \), the matrix \(-A^2 + RS + (AR + RA^T)X\) is stable. The uniqueness of \( X \) follows from the fact that the stable invariant subspace of \( H_X \) is unique. Part b) follows directly from the general Hamiltonian-Schur decomposition (Theorem 3.1 in [19]), since \( H_Y \) is similar to the complex Hamiltonian matrix

\[
\hat{H}_Y = \begin{bmatrix} -(A^2 + RS) & i(AR - RA^T) \\ -i(SA - A^T S) & (A^2 + RS)^T \end{bmatrix}.
\]

(15)

Note that Theorem 1 gives sufficient conditions for the existence of a stabilizing symmetric and skew-symmetric solution of the damped Riccati equations which are based on the eigenvalues of their associated Hamiltonian matrix. Therefore from this point forward we assume that solutions to the SDARE and the SSDARE exist, as is usually the case, and examine their properties. Continuing, the relationship between \( X \) and \( Y \) is explicitly derived in the following lemma.

**Lemma 2** Suppose there exist a stabilizing solution \( X \) to the SDARE and a stabilizing solution \( Y \) to the SSDARE. Then the following equations hold:

\[
S + A^T Y + X(A - RY) = 0, \\
S + A^T X - Y(A - RX) = 0.
\]

(16)

Proof. Since the matrix \( \begin{bmatrix} I & X \end{bmatrix} \) spans the stable invariant subspace of \( H_X \) and \( H_1H_X = H_Y H_1 \) then the matrix

\[
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} A & -R \\ S & A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}
\]

spans the stable invariant subspace of \( H_Y \). Thus, the stabilizing solution \( Y \) to the SSDARE may be written as

\[ Y = Y_2Y_1^{-1} = (S + A^T X)(A - RX)^{-1}. \]

The second formula in (16) follows by taking the conjugate transposition of the first.

**Lemma 3** Suppose \( X \) is a real symmetric solution of (12) which stabilizes \(-A^2 + RS\) and \( Y \) is a real skew-symmetric solution of (14) which stabilizes \(-A^2 - RS\). Then the products \(-(A - RX)(A - RY)\) and \(-(A - RY)(A - RX)\) are stable.
Proof. By examining the product $P^{-1}_X H X P_X$, with $P_X := \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$, and making use of the SDARE, we note that the product $P^{-1}_X H X P_X$ is upper block triangular with the upper left-hand block being $-A^2 + RS + (AR + RA^T)X$. If, however, we form the matrix $P^{-1}_X H X P_X = P^{-1}_X H_2 P_Y P^{-1}_Y H_1 P_X$ with $P_Y := \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$, then we obtain

$$P^{-1}_X H X P_X = \begin{bmatrix} -A + RY & R \\ S + A^T Y + X(A - RY) & X R + A^T \end{bmatrix} \times \begin{bmatrix} A - RX & -R \\ S + A^T X - Y(A - RX) & Y R + A^T \end{bmatrix}. \quad (17)$$

By Lemma 2, the lower left-hand blocks in both factors of (17) are zero. Thus, by examining the upper left-hand block of (17) and by noting that the eigenvalues of $-(A - RX)(A - RY)$ and $-(A - RY)(A - RX)$ are the same, we have the required result. \[ \square \]

The previous lemma illustrates that the product $-(A - RX)(A - RY)$ is stable. We need, however, that the products $-(A - RX)(A - RX)$ and $-(A - RY)(A - RX)$ are stable. This is equivalent to the statement that all the eigenvalues of $(A - RX)$ and $(A - RY)$, respectively, lie in the damped region of the complex plane. Note that in the following text we use the convention that $A \geq 0$ ($A > 0$) signifies that $A$ is a nonnegative definite (positive definite) matrix.

**Theorem 4** Suppose that $X$ is a real symmetric dampening solution of (12) and $Y$ is the real skew–symmetric dampening solution of (14). Then the closed–loop matrices $A - RX$ and $A - RY$ have eigenvalues within the closure of the union of the damped and anti-damped regions of the complex plane (excluding the point $0$).

**Proof.** The proof employs the standard result from Lyapunov stability theory [6, 17], namely that:

$$L \text{ real, stable } \iff \exists M = M^T > 0 \text{ such that } P < 0, \text{ where } P = L^T M + M L. \quad (18)$$

As a candidate for $M$ we take

$$M = X R X - A^T X - X A - S. \quad (19)$$

If $-(A - RX)^2$ is stable then the spectrum of $A - RX$ is in the damped region, so by (18), it suffices to show that

$$P = -(A - RX)^2 M - M(A - RX)^2 < 0, \quad M > 0. \quad (20)$$
By Lemma 2 we have $A^T X + S = Y (A - RX)$, and thus $M = -A^T X - X (A - RX) - S = -(X + Y)(A - RX)$. Further, by noting that $X = X^T$, $Y = -Y^T$, $M = M^T$, and $M = -(A - RX)^T (X - Y)$, it follows that


By using the fact that $A^T X + S = Y (A - RX)$, it is possible to further simplify $P$ to

$$P = -2(A - RX)^T (XRX + S)(A - RX) < 0. \quad (21)$$

Here we have assumed that $S$ is positive definite. If $S$ is nonnegative definite, a simple continuity argument may be made to achieve the same result, (see [10]). We have also made use of the fact that the matrix $-(A - RX)(A - RY)$ is stable, which implies in particular that $A - RX$ is non-singular.

Proceeding to the second proposition in (20), and using similar techniques as above we have


Since $Q$ is negative definite and $-(A - RY)(A - RX)$ is stable by construction, by (18) $M$ must be positive definite. Since $P$ is negative definite by (21), it follows that $-(A - RX)^2$ is stable and hence all eigenvalues of $A - RX$ are in the damped region of the complex plane. By applying the same techniques for the closed–loop matrix $A - RY$, and letting $M = Y RY + A^T Y - Y A + S$, the proof is complete. \qed

Note that we obtain that all the eigenvalues are in the closure of the damped and anti-damped regions excluding the point zero. If we want to ensure that all eigenvalues are in the interior of these regions, we can achieve this either by choosing $S > 0$ or by providing an appropriate stabilizability and detectability assumption. This is a well known result for standard stabilization problems and carries over in a canonical way.

**Remarks:** In general the feedbacks derived from $X$ and $Y$ will yield eigenvalues of the closed–loop system in the union of the stable–damped and the unstable–damped region. To move the poles from the unstable–damped region into the damped region one may solve a degenerate Riccati equation, (see [8]). This is a standard procedure with which to reflect the eigenvalues at the imaginary axis. If we compute the symmetric, stabilizing solutions of the following equations

$$\begin{align*}
(A - RX + \sigma I)^T U + U (A - RX + \sigma I) - URU &= 0 \quad (22) \\
(A - RY + \sigma I)^T V + V (A - RY + \sigma I) - Y RY &= 0, \quad (23)
\end{align*}$$

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then for $\sigma = 0$, the eigenvalues of $A - RX$ and $A - RY$ in the right half-plane will be reflected back across the imaginary axis into the damped region, becoming the eigenvalues of $A - R(X + U)$ and $A - R(Y + V)$ respectively. By setting $\sigma > 0$, a degree of stability $\sigma$ is afforded to the system, i.e., the system will be damped such that the real part of any closed-loop eigenvalue will be less than $-\sigma$.

Taking the sum of the symmetric solutions $X$ and $U$ of (12) and (22), one obtains the residual equation

$$S_{XU} := (X + U)R(X + U) - A^T(X + U) - (X + U)A.$$  \hspace{1cm} (24)

Once $S_{XU}$ has been formed, one can use a standard Algebraic Riccati Equation (4) to compute $W_{XU} = (X + U)$ directly. This provides a means of checking the sensitivity of the feedback via existing theorems concerning eigenvalue sensitivity of standard Riccati equations [13]. It also provides valuable information concerning the required state weighting to achieve a dampening controller.

In practice, the invariant subspaces from which the solutions $X$ and $Y$ of (12) and (14) may be computed are produced simultaneously via the periodic Schur algorithm [5, 9], which is a numerically robust method. Although the solution derived from $X + U$ seems to be more in line with standard theory than that from $Y + V$, numerical experiments seem to indicate that the non-symmetric sum $Y + V$ usually, but not always, produces feedback matrices of smaller norm. Furthermore, since $X$ and $Y$ may be produced simultaneously with little extra computational effort, and since the closed-loop eigenvalues of $A - R(X + U)$ and $A - R(Y + V)$ are different, it may be useful for the control system designer to have both solutions.

The eigenvalues of the closed-loop system may also be restricted to slimmer cones in the left half-plane. This is accomplished in part by replacing the Hamiltonian system $H_{X,2} := H_X = H_2H_1$ with systems of higher periodicity, namely

$$H_{X,p} = (-1)^pH_2H_1^{p-1}.$$  \hspace{1cm} (25)

The dampening solution $X$ from the Algebraic Riccati Equation associated with (25) is computed in analogous ways and may be used to produce feedbacks by which the eigenvalues of the resulting closed-loop system are contained within regions bounded by the stability cones that subtend the angle $180/\rho$ degrees. This may be done quite efficiently as the computational cost goes up linearly with the periodicity $p$, (see [5, 9]). Algorithmic details concerning this and other aspects of the Damped Algebraic Riccati Equations may be found in [10].

It should also be noted that by substituting $A$ with $A + \rho I$ in the Damped Algebraic Riccati Equations, the vertices of the cones of the damped region may be placed at the point $-\rho$ on the real line. This adds another degree of flexibility to the method of this paper.

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Finally another advantage of the new method that we have introduced compared with the method introduced by Kawasaki et. al. in [12] should not remain unmentioned. Not only has our method a complexity of order one less, but also by using the periodic Schur decomposition we avoid forming explicitly the product $HX$. In [12] such a product is explicitly formed. Explicitly forming these products usually increases the condition number of the problem, and this can lead to a loss of half the significant digits in the numerical computation. It is also worth noting that there is a one–to–one correspondence between the “damped-stability” region bounded by a hyperbola and parameterized by $m$ in [12] and the “damped-stability” region bounded by a blunted cone and parameterized by $\sigma$ in this paper.

3 Numerical Example

In this section we illustrate the results discussed in the previous sections via a numerical example. More numerical examples may be found in [10].

Example 1 In this example, we dampened a system of springs, dashpots, and masses with two inputs, as shown by Figure 2. The system is modeled by the

![Coupled Spring Experiment](image)

Figure 2: Coupled Spring Experiment

following time–invariant linear system

$$A = \begin{bmatrix} 0 & I \\ M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \tilde{B} \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix},$$

(26)

where $M = \mu I$, $D = \delta I$,

$$K = \kappa \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix},$$

(27)
In this experiment, we demonstrate the efficiency of the Damped Algebraic Riccati Equations in producing dampening controllers for systems of large dimension. In this case, \( n = 60 \) (thirty springs), \( \mu = 4 \), \( \kappa = 1 \), and \( \delta = 4 \). The eigenvalues of the closed–loop matrices \( A - R(X + U) \) and \( A - R(Y + V) \) are shown in Figure 3.

![Figure 3: Damped Eigenvalues](image)

4 Conclusion

In this paper we have proposed a new method to produce a dampening controller. It promises to be an efficient and numerically reliable method to restrict the eigenvalues of a closed–loop state matrix to relatively elaborate regions in the left half–plane, without resorting to pole–placement.

Still, many issues remain open, and are presently being investigated. Among these are standard analysis of the sensitivity of the eigenvalues of the closed–loop state matrix, scalings of the Hamiltonian to produce optimal results. Also, we are aware of different parameterized formulations of the periodic Hamiltonian system in complex arithmetic which also produce excellent dampening controllers. We have also observed experimentally that convex–combinations of the feedbacks discussed
in this paper remain stable. These results will be given elsewhere.

References


