TECHNISCHE UNIVERSITÄT BERLIN SS07
Fakultät II - Institut für Mathematik
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5. Übung Visualisierung in der Mathematik

(Matrix representations of $Isom(E^2)$)

Übungsaufgaben

Information: The course web site is http://www.math.tu-berlin.de/geometrie/Lehre/SS07/MathVis/. All assignments will be posted there in .pdf format. You can also check the web site for contact information for the teachers, and other information related to this class.

1. Aufgabe

Reflections A reflection in E^2 is an isometry which fixes a line pointwise and sends other points to their mirror image on the other side of the line. Suppose the fixed line l is given by the line equation $ax + by + c =$ and suppose further that $a^2 + b^2 = 1$. In this exercise we'll deduce the general form for the matrix representing this reflection; we'll call it R_l.

- What allows us to assume that $a^2 + b^2 = 1$?
- When $c = 0$, the line goes through the origin $(0, 0)$. Show that in this case, the matrix form for the representation is $R_l = \begin{pmatrix} 1-2a^2 & -2ab & 0 \\ -2ab & 1-2b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [Hint: Show that $\{e_1, e_2\} = \{(b, -a), (a, b)\}$ is an orthornormal basis for E^2. Show that in this basis the matrix for R_l is $R_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Finally, conjugate by the change-of-coordinate matrix to show that the matrix form for the standard basis has the desired form.]
- Show that in the general case ($c \neq 0$), the matrix form is: $R_l = \begin{pmatrix} 1-2a^2 & -2ab & -2ac \\ -2ab & 1-2b^2 & -2bc \\ 0 & 0 & 1 \end{pmatrix}$. [Hint: Show that the point $P := (-ac, -bc)$ lies on the line l. Let T be the translation taking P to the origin $(0, 0)$. Considering T as a change of coordinates, show that in the new coordinate system l has equation $ax + by = 0$ and so the reflection has the form deduced above. Finally, conjugate this matrix by T to show that the matrix for the reflection in the standard coordinate system has the desired form.]
- Calculate the matrix form for the reflection $R \in Isom(\mathbb{E}^2)$ defined by reflection in the the following lines: $x = 0$, $y = 2$, $x + 2y = 3$, and the line going through the points $P = (1, 1)$ and $Q = (0, 4)$. (That's four difference matrices).
- Generalize this result to reflections in planes in 3 dimensions, and calculate the matrix for the reflections in the following planes: $x = 3$, and $x + y + z = 0$.

2. Aufgabe

Rotations We have seen that rotations of E^3 can be expressed using unit quaternions. To be exact, if $h = a + bi + cj + dk$ is a unit quaternion, and $p = xi + yj + zk$ is a purely imaginary quaternion (representing a variable point $(x, y, z) \in E^3$), then the quaternion defined by $q := hp\overline{h}$ is an imaginary quaternion which represents the result of rotating p around the axis with direction (b, c, d) through an angle equal to $2 * cos^{-1}(a)$. What does

the 4x4 matrix R_h for this rotation look like in terms of the coordinates (a, b, c, d) of the unit quaternion?
By laboriously multiplying the above expression out, one arrives at the following:

$$R_h = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) & 0 \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(-ab+cd) & 0 \\ 2(-ac+bd) & 2(ab+cd) & a^2-b^2-c^2+d^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Apply the above formula to calculate the form for the following rotations:
  - Rotation of π around the y-axis.
  - Rotation of $120°$ around the axis $(1, 1, 1)$
  - Rotation through an angle α around the z-axis.
- Describe how you might verify that the formula actually represents the desired rotation.

3. Aufgabe

Consider the three reflections given by reflecting in the coordinate planes $x = 0$, $y = 0$, and $z = 0$.

- Calculate the matrices for these reflections as 4x4 matrices.
- Calculate the rotation matrices gotten by taking the products of these reflections, two at a time. How many different rotations are they? Describe their geometrical significance.
- What other matrices can be gotten by taking further products of the reflections and rotations obtained so far?
- Show that the set of all such matrices forms a group of order 8.

4. Aufgabe

Consider a regular tetrahedon inscribed in a unit cube (i.e., its four vertices are $\{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$. In this exercise you'lll calculate the orientation-preserving symmetries of this tetrahedron.

- Show there are rotation axes of order 3 around axes passing through the vertices
- Show that there are rotations of order 2 around axes passing through the midpoints of the edges
- Show that, including the identity symmetry, there are 12 orientation-preserving symmetries of the tetrahedron and argue that these form a group.
- Calculate the matrices for this group.
- **[Optional]** Show that the tetrahedron has mirror plane symmetry around the planes which bisect the edges perpendicularly. This gives 6 reflections. We know that the full group of symmetries of the tetrahedron has 24 elements (twice the number of orientation-preserving ones). What are the other 6 orientation-reversing isometries?