TUTORIAL

In the following, let \( f(z) = \frac{az+b}{cz+d} \) be a Möbius transformation. A circle means either an ordinary circle or a line.

**Exercise 1**

1. Show that the Möbius transformation \( f(z) = \frac{1}{z} \) maps circles to circles. Deduce the same for all Möbius transformations.
2. Determine \( f \) such that \( f(0) = i, f(i) = \infty, \) and \( f(\infty) = 1. \) Draw the images of the four quadrants. What happens to the coordinate lines?

**Exercise 2**

1. Let \( A \in SL(2, \mathbb{C}) \) be the matrix associated to \( f. \) Show that \( f \) maps the upper half-plane onto itself \( \iff A \in SL(2, \mathbb{R}). \)
2. Let \( n \) be the number of fixed points of such an \( f \neq Id \) on the real axis. \( f \) is said to be hyperbolic if \( n = 2, \) elliptic if \( n = 0, \) and parabolic if \( n = 1. \) Show that
\[
tr(A) \begin{cases} > 2 & \iff f \text{ is hyperbolic} \\ = 2 & \iff f \text{ is parabolic} \\ < 2 & \iff f \text{ is elliptic} \end{cases} \quad (1)
\]

**Exercise 3**

1. Show that every Möbius transformation has at least one fixpoint.
2. Consider a Möbius transformation \( f \) with exactly one fixpoint. Show that there exists a Möbius transformation \( g \) and \( b \in \mathbb{C} \) such that
\[
(g^{-1} \circ f \circ g)(z) = z + b.
\]
Homework

Exercise 1 5 points
Let $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ be the unit disc and $\mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im } z \geq 0 \}$ be the upper half-plane. Find a Möbius transformation $f : \mathbb{C} \to \mathbb{C}$ with

$$f(0) = -1, \quad f(i) = 0, \quad \text{and } f(\infty) = 1.$$ 

Show that $f(\mathbb{H}) = \mathbb{D}$.

Exercise 2 10 points
Let $a, b, c, d \in \mathbb{C}$ and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $f(z) = \frac{az + b}{cz + d}$ a Möbius transformation.

1. Determine $a, b, c, d$ if $f$ has at least 3 fixpoints.
2. When does $f$ have exactly two fix points?
3. Let $g$ be a Möbius transformation with fix points 0 and $\infty$. Show that there exists $s \in \mathbb{C} \setminus \{0, 1\}$ such that $g(z) = sz$.
4. Let $f$ be a Möbius transformation with two disjoint fix points in $\hat{\mathbb{C}}$. Show that there exists a Möbius transformation $\tau$ and $s \in \mathbb{C} \setminus \{0, 1\}$ such that

$$(\tau^{-1} \circ f \circ \tau)(z) = sz$$

5. Let $f(z) = sz$ with $s \in \mathbb{C} \setminus \{0, 1\}$ and $z_0 \in \mathbb{C} \setminus \{0\}$. Let $(z_n)_{n \in \mathbb{N}}$ be defined by $z_n = f \circ \cdots \circ f(z_0)$. Determine the behavior (convergence, divergence, periodicity) of the sequence $(z_n)$ depending on the parameter $s$.

Exercise 3 5 points
Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ four disjoint points on a circle as shown in the figure on the right.

a) Show that the cross-ratio $\text{cr}(z_1, z_2, z_3, z_4)$ is negative, whereas the cross-ratio $\text{cr}(z_1, z_3, z_2, z_4)$ is positive.
b) Verify that $|\text{cr}(z_1, z_2, z_3, z_4)| + 1 = |\text{cr}(z_1, z_3, z_2, z_4)|$ and deduce the following equality for the lengths of the edges and the diagonals of the circular quadrilateral:

$$|z_1 - z_3| |z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$ 

Hint: Apply a suitable Möbius transformation to the circle that maps it to the real axis.