Complex Analysis - Exercise Sheet 8
- Analytic continuation, fundamental group, homology -

due 13-14.06.2012

Tutorial

Exercise 1
Let \([-i, i] = \{ z \in \mathbb{C} | z = ix, x \in [-1, 1] \subset \mathbb{R} \}\) and \(G = \mathbb{C} \setminus [-i, i]\). Let \(z_0 = 1\) and \(\gamma_z\) an arbitrary curve in \(G\) from \(z_0\) to \(z\). Prove that:

\[
F : G \to \mathbb{C}, \quad F(z) = \int_{\gamma_z} \frac{1}{1 + \zeta^2} d\zeta
\]

is independent of the choice of \(\gamma_z\) and is holomorphic.

Exercise 2
Let \(c : [0, 1] \to \mathbb{C}\) be a closed curve with only finitely many self-intersections. Let \(p = c(t_0)\) be a regular point of the curve and \(\gamma : [-\varepsilon, \varepsilon] \to \mathbb{C}\) a differentiable curve intersecting \(c\) in \(p\) transversally, i.e. \(\det(\dot{c}(0), \dot{\gamma}(t_0)) \neq 0\). If \(a = \gamma(-\varepsilon)\) and \(b = \gamma(\varepsilon)\), how do \(n(c, a)\) and \(n(c, b)\) differ? How can you use this to calculate the index of an arbitrary point?

Exercise 3
“Dog on the leash”-Theorem.
Informal: How do you prevent your dog from winding around a tree? Well – keep him on a leash shorter than the distance to the closest tree.
Mathematical: Let \(c_1, c_2 : [0, 1] \to \mathbb{C}\) be two closed curves and \(z_0 \in \mathbb{C} \setminus (|c_1| \cup |c_2|)\). Further assume that for all \(t \in [0, 1]::

\[
|c_1(t) - c_2(t)| < |c_1(t) - z_0|.
\]

Show that \(n(c_1, z_0) = n(c_2, z_0)\). Calculate \(n(c, 0)\) for the following curves \(c : [0, 2\pi] \to \mathbb{C}, \varphi \mapsto e^{ni\varphi} + re^{ki\varphi}\), where \(n, k \in \mathbb{Z}, r \in (0, \infty), r \neq 1\).
Homework

Exercise 1 12 points
Let $G$ and $F : G \to \mathbb{C}$, $F(z) = \int_{\gamma_z} \frac{1}{1+\zeta^2}d\zeta$ as in Tutorial Exercise 1. Let $g : G \to \mathbb{C}$, $g(z) := \frac{\pi}{4} + F(z)$.

a) Let $\gamma_1, \gamma_2 : [0, 2\pi] \to \mathbb{C}$, $\gamma_1(t) = 2e^{it}$, and $\gamma_2(t) = re^{it} + i$ with $r \in (0, 2)$.

Show that:

$$\int_{\gamma_1} \frac{1}{1 + \zeta^2} d\zeta = 0 \quad \text{and} \quad \int_{\gamma_2} \frac{1}{1 + \zeta^2} d\zeta = \pi.$$  

b) Is it possible to extend $g$ onto a region $H$ with $G \subseteq H$?

Consider $g(z + \varepsilon)$ and $g(z - \varepsilon)$ for $z \in [-i, i]$ and $\varepsilon > 0$ and use part a).

c) Consider $x > 0$ in $\mathbb{R}$ and show that $(\tan)'(x) = 1 + \tan^2(x)$ and $g(x) = \arctan(x)$ resp. $\tan(g(x)) = x$.

Hint: Calculate $F(x)$ by explicit integration.

d) Show that $\tan(g(z)) = z$ for all $z \in G$.

Hint: Use the identity theorem and part c).

Exercise 2 3 points
Let $c$ be a simple closed piecewise continuously differentiable curve. Show that $\mathbb{C} \setminus |c|$ has at least two components.

Exercise 3 5 points
Let $U \subset \mathbb{C}$ open and bounded such that $\mathbb{C} \setminus U$ is connected. Show that all closed continuous curves in $U$ are homologous to a constant curve, i.e. null-homologous.