Complex Analysis - Exercise Sheet 9

- Residue Theorem, Rouché’s Theorem -

due 20-21.06.2012

Tutorial

Exercise 1
Let $G \subset \mathbb{C}$ be a region, $a \in G$ and $f : G \setminus \{a\} \to \mathbb{C}$ a holomorphic function.

a) If $f$ has a pole of order one in $a$ then

$$\text{res}(f, a) = \lim_{z \to a} (z - a)f(z).$$

b) If $f$ has a pole of order $k$ in $a$ then

$$\text{res}(f, a) = \lim_{z \to a} \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial z^{k-1}}((z - a)^k f(z)).$$

c) Calculate $\text{res}(\frac{\cot z}{z^2(z+1)}, \pi)$.

Exercise 2
Let $f, g : G \to \mathbb{C}$ be two holomorphic functions. Can you express the residue of $h = f \cdot g$ at $z_0$ in terms of derivatives of $f$ and $g$? What happens if $g$ has a root of order 2 at $z_0$?

Exercise 3
Calculate the integral $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}$ using the residue theorem.

Exercise 4
Determine the number of roots (including multiplicity) of the polynomial $p(z) = z^8 - 5z^3 + z - 2$ contained in the unit disc.
Exercise 1 6 points
Calculate the following integrals:

(i) \[ \int_{0}^{\infty} \frac{x^2}{x^4 + 6x^2 + 5} \, dx, \]
(ii) \[ \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx \text{ with } a \in \mathbb{R}, a > 0. \]

Exercise 2 4 points
Calculate \( \text{res}(e^{z+1/z}, 0) \).

Exercise 3 10 points
Let \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

a) Determine the number of roots of the following polynomial in the unit disk (including their multiplicity):
\( p(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1. \)

b) Let \( U \subset \mathbb{C} \) be an open set containing \( \overline{\mathbb{D}} \). Let \( f : U \to \mathbb{C} \) be a holomorphic function with \( f(\overline{\mathbb{D}}) \subset \mathbb{D} \). Show that \( f \) has exactly one fixpoint in \( \mathbb{D} \).

c) Let \( \lambda \in \mathbb{R} \) with \( \lambda > 1 \). Show that the equation \( e^{z-\lambda} = z \) has exactly one solution in \( \mathbb{D} \). Show further that this solution is real.