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Geometry of the Hopf Mapping and Pinkall's Tori of Given Conformal Type

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One of the most fertile geometrical examples in mathematics is the Hopf mapping from the 3-sphere S^3 to the 2-sphere S^2. On the occasion of the Conference on Computers in Geometry and Topology at the University of Illinois at Chicago, we considered three aspects of this mapping which are particularly well suited to investigation by computer graphics.

The study of Hamiltonian dynamical systems can be motivated and illustrated by a linear system which leads to the Hopf fibers as circular orbits lying on a constant energy 3-sphere. This aspect has been reported at length elsewhere in the article of Hüseyin Koçak, Fred Bisshopp, David Laidlaw, and the author [1].

Regular polytopes in 4-space can be decomposed into rings of polyhedra which correspond to solid tori which are preimages of cells on the 2-sphere under the Hopf mapping. This aspect has appeared in the author's article in the proceedings of the conference Shaping Space [2].

The third topic considered was an elementary presentation of a remarkable construction by Ulrich Pinkall which determines the conformal structures of tori obtained by lifting a closed curve on S^2 under the Hopf mapping. This short note is an exposition of Pinkall's result which is well-suited to interactive computer graphics investigation.

In [3], Pinkall showed that the inverse image under the Hopf mapping of a simple closed curve on S^2 is a flat torus on S^3 which is conformally equivalent to a parallelogram in the plane with basis vectors

(0,1) and (L/2,A/2), where L is the length of the curve and A is the (oriented) area it encloses on S^2. Stereographic projection then yields embedded tori in R^3 of the same conformal type. This result gives an explicit solution to a problem first solved by Adriano Garsia using methods which were nonconstructive and not suited for the production of examples.

Pinkall's elegant approach utilizes quaternionic multiplication on S^3, arc length parametrization of a curve on S^2, choice of a lifting of the curve which is orthogonal to the Hopf fibers, and calculation of the area by integration of a curvature form on a circle bundle. His article is illustrated by computer-generated line images. In order to investigate these Hopf tori by means of interactive computer graphics, it seems more convenient to use Cartesian coordinates, arbitrary parametrization of the base curve, a specific lift not necessarily orthogonal to the fibers, and an explicit calculation of the area. Fortunately it is possible to carry out Pinkall's argument without introducing any of the technical apparatus of modern differential geometry, thereby producing a proof which, though less elegant, is more elementary, and accessible to students with a knowledge of the classical geometry of curves and surfaces. In this note we give such a direct presentation, together with a set of illustrations indicating the way the stereographic projections of Hopf tori transform under inversions.

We will only need curves on the 2-sphere which are polar coordinate function graphs, $X(\theta) = (\cos(\theta)\sin(\phi(\theta)), \sin(\theta)\sin(\phi(\theta)), \cos(\phi(\theta)))$, where $0 \leq \theta \leq 2\pi$ and where $\phi(\theta)$ is a differentiable function of θ with period 2π and $0 \leq \phi(\theta) \leq \pi$. (Note that these are the coordinates used in astronomy, with ϕ measured down from the North Pole, rather than standard geographical coordinates measuring latitude up and down from the Equator.)

The velocity vector of the curve is given by

$$X'(\theta) = (-\sin(\theta)\sin(\phi(\theta)) + \cos(\theta)\cos(\phi(\theta))\phi'(\theta),$$
$$\cos(\theta)\sin(\phi(\theta)) + \sin(\theta)\cos(\phi(\theta))\phi'(\theta), -\sin(\phi(\theta))\phi'(\theta))$$

Thus the length L(t) is the integral of $\sqrt{X_\theta \cdot X_\theta} = \sqrt{[\sin^2(\phi(\theta)) + (\phi'(\theta))^2]}$ from 0 to t.

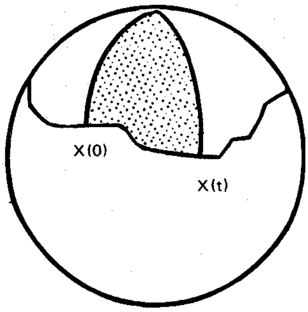

Figure 1

The area $A(t)$ of the wedge from the North Pole down to $X(0)$, along the curve to $X(t)$, and back up to the pole, is the integral of the area element (see Figure 1)

$$|X_\theta \times X_\phi| = \sin(\phi)$$

so

$$A(t) = \int_0^t \int_0^{\phi(\theta)} \sin(\phi)d\phi d\theta = \int_0^t (1 - \cos(\phi(\theta))d\theta$$

To parametrize the unit 3-sphere S^3, we introduce "toral" coordinates $Z(\alpha,\beta,\gamma) = (\cos(\alpha)\sin(\gamma),\sin(\alpha)\sin(\gamma),\cos(\beta)\cos(\gamma),\sin(\beta)\cos(\gamma))$, where $0 \leq \alpha,\ \beta \leq 2\pi$ and $0 \leq \gamma \leq \pi/2$. The Hopf mapping sends a point (x, y,u,v) in R^4 to the point $H(x,y,u,v) = (2xu + 2yv, -2xv + 2yu, u^2 + v^2 - x^2 - y^2)$ in R^3. Thus $H(Z(\alpha,\beta,\gamma)) = (\cos(\alpha - \beta)\sin(2\gamma), \sin(\alpha - \beta)\sin(2\gamma), \cos(2\gamma))$, and the image of S^3 under H is the 2-sphere S^2. The preimage of any point $(\cos(\theta)\sin(\phi),\sin(\theta)\sin(\phi), \cos(\phi))$ is then an entire curve $Z(\theta + \psi,\psi,\phi/2) = (\cos(\theta + \psi)\sin(\phi/2), \sin(\theta + \psi)\sin(\phi/2),\cos(\psi)\cos(\phi/2),\sin(\psi)\cos(\phi/2))$, where $0 \leq \psi \leq 2\pi$. It is easy to show that the preimage of any point is a great circle on S^3. If $X(\theta)$ is a closed curve on S^2 defined by a function $\phi(\theta)$, then these circles fit together to form a torus in S^3, called a Hopf torus. For example, if the curve $X(\theta)$ is given by the function $\psi(\theta) = 2c$, a constant, then the Hopf torus is $(\cos(\theta + \psi)\sin(c),\sin(\theta + \psi)\sin(c),\cos(\psi)\cos(c), \sin(\psi)\cos(c))$, a product torus $Z(\theta + \psi,\psi,2c)$ on S^3.

Some of the motivation for the Hopf mapping becomes clearer if we introduce complex coordinates on S^3, writing a 4-tuple $(x,y,u.v)$ as a pair $[x + iy, u + iv] = [z,w]$ of complex numbers. The Hopf mapping is then defined by sending the pair $[z,w]$ to the ratio w/z if $z \neq 0$ and ∞ if $z = 0$, thus giving a point on C with ∞ adjoined. Inverse stereographic projection (plus a reflection) then takes w/z to $[2z\bar{w}, w\bar{w} - z\bar{z}]$ and takes ∞ to $[0,1]$. In polar coordinates, $Z(\alpha,\beta,\gamma) = [\sin(\gamma)e^{i\alpha}, \cos(\gamma)e^{i\beta}]$, and $H[\sin(\gamma)e^{i\alpha}, \cos(\gamma)e^{i\beta}] = [\sin(2\gamma)e^{i(\alpha-\beta)}, \cos(2\gamma)]$.

The claim is that any Hopf torus is flat, with Gaussian curvature identically zero. The easiest way to establish this is to show that this surface is isometric to a specific region in the plane. We first compute the metric coefficients of the surface $Y(\theta,\psi) = Z(\theta + \psi, \psi, \phi(\theta)/2)$:

$$\begin{aligned} Y_\theta = (&-\sin(\theta + \psi)\sin(\phi(\theta)/2) + \cos(\theta + \psi)\cos(\phi(\theta)/2)\phi'(\theta)/2, \\ &\cos(\theta + \psi)\sin(\phi(\theta)/2) + \sin(\theta + \psi)\cos(\phi(\theta)/2)\phi'(\theta)/2, \\ &-\cos(\psi)\sin(\phi(\theta)/2)\phi'(\theta)/2, \\ &-\sin(\psi)\sin(\phi(\theta)/2)\phi'(\theta)/2), \\ Y_\psi = (&-\sin(\theta + \psi)\sin(\phi(\theta)/2), \cos(\theta + \psi)\sin(\phi(\theta)/2), \\ &-\sin(\psi)\cos(\phi(\theta)/2), \cos(\psi)\cos(\phi(\theta)/2)) \end{aligned}$$

Then $g_{11} = Y_\theta \cdot Y_\theta = \sin^2(\phi(\theta)/2) + (\phi'(\theta)/2)^2$, $g_{12} = Y_\theta \cdot Y_\psi = \sin^2(\phi(\theta)/2)$, and $g_{22} = Y_\psi \cdot Y_\psi = 1$.

We now define a planar region by $W(\theta,\psi) = (L(\theta)/2, A(\theta)/2 + \psi)$. Then $W_\theta = (L'(\theta)/2, A'(\theta)/2), W_\psi = (0,1)$. The metric coefficients are then $G_{11} = W_\theta \cdot W_\theta = L'(\theta)^2/4 + A'(\theta)^2/4$, $G_{12} = W_\theta \cdot W_\psi = A'(\theta)/2$, and $G_{22} = W_\psi \cdot W_\psi = 1$. First note $G_{22} = g_{22}$. Since $A'(\theta)/2 = [1 - \cos(\phi(\theta))]/2 = \sin^2(\phi(\theta)/2)$, we also have $G_{12} = g_{12}$. Finally, $L'(\theta)^2/4 + A'(\theta)^2/4 = \sin^2(\phi(\theta))/4 + \phi'(\theta)^2/4 + \sin^4(\phi(\theta)/2) = \sin^2(\phi(\theta)/2)\cos^2(\phi(\theta)/2) + \phi'(\theta)^2/4 + \sin^2(\phi(\theta)/2)\sin^2(\phi(\theta)/2)$, so $G_{11} = g_{11}$. Therefore the Hopf torus has precisely the same metric coefficients as a region in the plane bounded by two vertical segments of length 2π and two other curves which are parallel translates of one another. This identification space is isometric to the parallelogram with vertices at $(0,0)$, $(L/2, A/2)$, $(L/2 + 2\pi, A/2)$, and $(0, 2\pi)$ (see Figure 2).

Pinkall shows that it is possible to choose curves on the sphere which yield all possible parallelograms with third vertex in a given fun-

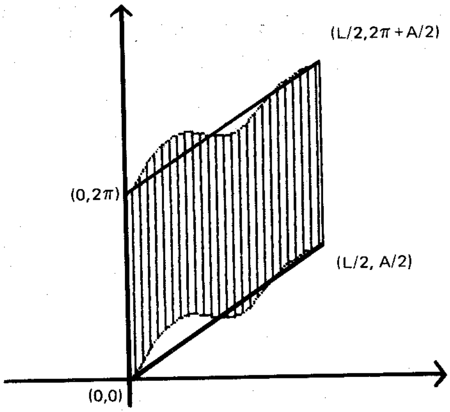

Figure 2

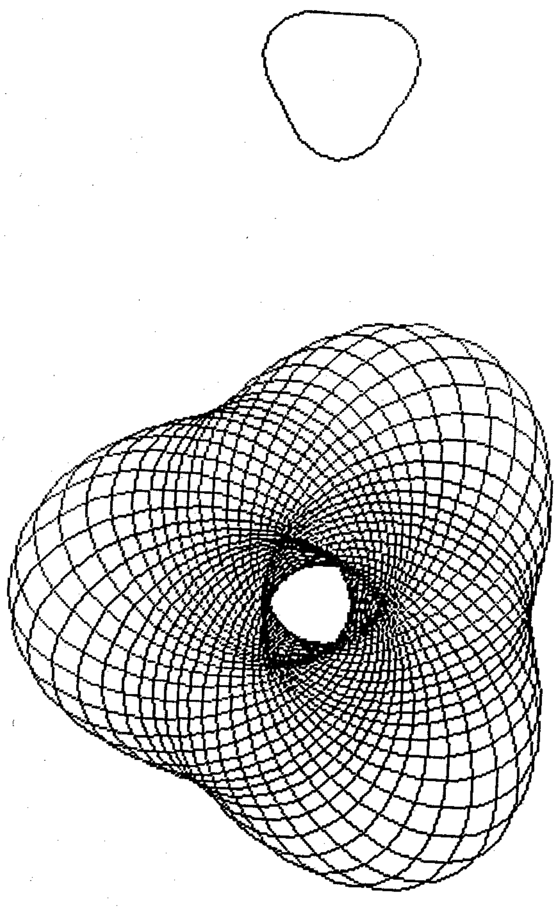

Figure 3

damental region, so this completely solves the problem of finding tori on the 3-sphere of given conformal types. Stereographic projection gives embedded tori in ordinary 3-space with the same conformal types (see Figure 3).

Added in Proof: Joel Weiner has been able to use the techniques of this paper to extend his results on Flat Tori in S^3 and Their Gauss Maps.

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