Prof. John Sullivan
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Homework Assignment 3

Homework due May 6th at the end of the lecture.

Topics: Isoperimetric inequality, Cauchy–Crofton formula, Four vertex theorem, Support function

Problem 1

Let $\overline{AB}$ be a line segment in $\mathbb{R}^2$. Let γ be the curve from A to B of fixed length $L > \text{len}(\overline{AB})$ such that γ joined to $\overline{AB}$ is a simple closed curve enclosing maximal area. Prove that γ is a circular arc.
(4 points)

Problem 2

Consider a convex simple closed plane curve α. The *support function* h gives the (signed) distance of the tangent line at each point of α to the origin.

a) For every value $\phi \in [0, 2\pi)$, there is a point on α with tangent vector in direction $(\cos(\phi), \sin(\phi))$. Given the support function $h(\phi)$ that gives the signed distance to this point for each ϕ, find a parametrization $\alpha(\phi)$ of the curve.
Hint: The value $h(\phi)$ directly gives a line along which this point lies; the exact point depends also on $h'(\phi)$.

b) Compute a formula for the curvature of $\alpha(\phi)$.

(2.5 + 1.5 points)

Problem 3

Conider a regular n-gon P_n with side length 1.

a) Compute its minimum and maximum width.

b) Compute its length using the Cauchy–Crofton formula.

(1.5+1.5 points)

Problem 4

The following curve is an example of a *trefoil knot* (*Kleeblattknoten*):

$$\alpha(t) = (4\cos(2t) + 2\cos(t), 4\sin(2t) - 2\sin(t), \sin(3t)), \quad 0 \le t \le 2\pi.$$

(Intuitively, a simple closed curve in $\mathbb{R}^3$ is a *knot* provided it cannot be smoothly deformed (always remaining simply closed) until it becomes a circle. A theorem of Fáry and Milnor asserts that every knot has total curvature strictly greater than 4π.

Problem 4 continued on next page…

a) Plot $\alpha(t)$ using a program of your choice. (See additional material on course website if you don't know what to use.)

b) Consider the orthogonal projection of α to the xy plane. Show that the resulting curve has total curvature exactly 4π. (This curve is not simply closed, and can therefore not be a knot).

c) Show that α can be deformed to a knot that has (numerically estimated) total curvature less than 4.01π.

(1.5 +1.5+2 points)