Exercise Sheet 1

Exercise 1: Scale rotations and complex numbers. (6 pts)
Consider the set of scale rotations in $\mathbb{R}^2$

$$C := \mathbb{R} \cdot \text{SO}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$ 

Together with addition and matrix multiplication $C$ becomes a field (note that each two elements of $C$ commute).

(i) Show that any linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ belongs to $C \setminus \{0\}$ if and only if $\det A > 0$ and there is some $r > 0$ such that

$$\langle Ax, Ay \rangle = r \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^2.$$ 

(ii) Show that the map $\Phi : \mathbb{C} \to C$, $a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

is an isomorphism of fields.

Exercise 2: Complex numbers and plane geometry. (7 pts)

(i) For which $a, b, c, d \in \mathbb{C}$ does the complex equation

$$azz \bar{z} + bz + c\bar{z} + d = 0$$ 

represent a line, for which a circle?

(ii) Write the equation of an ellipse centered at the origin,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{for some } a, b > 0,$$

in complex form.

(iii) Using complex numbers, prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.
Exercise 3: Cauchy-Riemann equations and differentiability. (7 pts)

(i) Show that the functions $f, g : \mathbb{C} \rightarrow \mathbb{C}$ given by
\[
  f(x + iy) := -4xy + 2i(x^2 - y^2 + 3) \\
  g(x + iy) := x^2 + y^2 + 2ixy
\]
are real differentiable, and check which of them satisfies the Cauchy-Riemann equations, i.e. is holomorphic.
Express both functions in terms of $z$ and $\bar{z}$ only.

(ii) Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by
\[
  f(x + iy) := \begin{cases} 
    xy^2(x+iy) & \text{for } z \neq 0 \\
    0 & \text{for } z = 0
  \end{cases}
\]
Show that $\lim_{z \to 0} \frac{f(z) - f(0)}{z}$ is equal to 0 if $z \to 0$ along any straight line, and it is equal to $\frac{1}{2}$ if $z \to 0$ along the curve $x = y^2$.
Conclude that $f$ satisfies the Cauchy-Riemann equations at $z = 0$, but it is not differentiable at $z = 0$ (neither complex nor real).

Due Tuesday, May 05, before the lecture.