Exercise Sheet 3

Exercise 1: Stereographic projection. (6 pts)
Let $\sigma : S^2 \to \hat{\mathbb{C}}$ be the stereographic projection. Every $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can be uniquely identified with a map $\tilde{f} : S^2 \to S^2$, given by
$$\tilde{f} := \sigma^{-1} \circ f \circ \sigma.$$ 
Determine $\tilde{f}$ for
(i) $f(z) = \bar{z}$, (ii) $f(z) = \frac{1}{z}$, (iii) $f(z) = -z$, (iv) $f(z) = \frac{1}{z}$
and describe the action of $\tilde{f}$ on $S^2$.

Bonus: (4 pts)
Find the most general form of $f$ for which $\tilde{f}$ is a rotation of $S^2$.

Exercise 2: Circular quadrilaterals. (4 pts)
Let $a, b, c, d \in \mathbb{C}$ be four distinct points on a circle in cyclic order. Prove the following statements using the complex cross ratio:
(i) $\angle(a, c, b) = \angle(a, d, b)$
(ii) $|a - c| |b - d| = |a - b| |c - d| + |b - c| |a - d|

Hint: Show $\text{cr}(a, b, c, d) + \text{cr}(a, c, b, d) = 1$ to prove (ii).

Exercise 3: Finding Möbius transformations. (4 pts)
For which $a, b, c, d \in \mathbb{R}$ does there exist a Möbius transformation that maps the points $(0, 1, 1 + i, i)$ to the points $(a + ib, c + ib, c + id, a + id)$. Find this map.

Exercise 4: Fixpoints of Möbius transformations. (6 pts)
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Prove the following statements:
(i) If $f$ has exactly one fixpoint, then there exists a Möbius transformation $g$ and $b \in \mathbb{C} \setminus \{0\}$ such that $(g^{-1} \circ f \circ g)(z) = z + b$.
(ii) If $f$ has two distinct fixpoints, then there exists a Möbius transformation $g$ and $a \in \mathbb{C} \setminus \{0\}$ such that $(g^{-1} \circ f \circ g)(z) = az$.

Due Tuesday, May 12, before the lecture.