TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis - SS 2015

http://www3.math.tu-berlin.de/geometrie/Lehre/SS15/ComplexAnalysis/

Exercise Sheet 4

Exercise 1: Classification of Möbius transformations. (8 pts)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, $f \neq id$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ a corresponding (normalized) 2×2-matrix.

(i) Show that

$$\begin{aligned} f \text{ parabolic } &:\Leftrightarrow\ a + d = \pm 2 \\ &\Leftrightarrow\ f \text{ has exactly one fixpoint.} \end{aligned}$$

(ii) Otherwise f has exactly two fixpoints and there exists a Möbius transformation g and $m \in \mathbb{C} \setminus \{0, 1\}$ such that (*cf. Exercise Sheet 03, Exercise 04*)

$$\left(g \circ f \circ g^{-1}\right)(z) = mz.$$

Show that

$$\begin{aligned} f \text{ elliptic } &:\Leftrightarrow\ a + d \in \mathbb{R} \quad \text{and} \quad |a + d| < 2 \\ &\Leftrightarrow\ m = e^{i\varphi} \neq 1 \quad \text{for some } \varphi \in \mathbb{R} \\ f \text{ hyperbolic } &:\Leftrightarrow\ a + d \in \mathbb{R} \quad \text{and} \quad |a + d| > 2 \\ &\Leftrightarrow\ m = \rho \neq 1 \quad \text{for some } \rho > 0. \end{aligned}$$

Hint: Show that $m = \lambda^2$ where λ satisfies $\lambda + \frac{1}{\lambda} = a + d$.

(iii) Consider the automorphism of the unit disc

$$g(z) := \frac{z - q}{\bar{q}z - 1}$$

for some $q \in \mathbb{C}$, $|q| < 1$. Show that g is elliptic and compute m.

Exercise 2: Möbius transformations with prescribed boundary. (6 pts)

(i) Show that the most general Möbius transformation of the upper half-plane to the unit disc has the form

$$f(z) = e^{i\varphi}\frac{z-a}{z-\bar{a}}$$

where $a \in \mathbb{C}$, $\mathrm{Im}(a) > 0$ and $\varphi \in [0, 2\pi)$.

(ii) Use (i) to show that the most general Möbius transformation of the unit disc to the upper half plane has the form

$$g(z) = \frac{\bar{a}z - ae^{i\varphi}}{z - e^{i\varphi}}.$$

Exercise 3: Two circles under a Möbius transformation. (6 pts)

Let $C_1, C_2 \subset \mathbb{C}$ be two circles. Prove the following statements:

(i) C_1 and C_2 intersect in two points if and only if there exists a Möbius transformation f such that $f(C_1)$ and $f(C_2)$ are two intersecting lines.

(ii) C_1 and C_2 touch if and only if there exists a Möbius transformation f such that $f(C_1)$ and $f(C_2)$ are two parallel lines.

Bonus: (2 pts)

Where does the image of the center of a circle under a Möbius transformation lie?

Due Friday, May 15, before the lecture.