Exercise 1: Classification of Möbius transformations. (8 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be a Möbius transformation, $f \neq id$, and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL(2, \mathbb{C})$ a corresponding (normalized) $2 \times 2$-matrix.

(i) Show that $f$ parabolic $\iff a + d = \pm 2$ $\iff f$ has exactly one fixpoint.

(ii) Otherwise $f$ has exactly two fixpoints and there exists a Möbius transformation $g$ and $m \in \mathbb{C} \setminus \{0, 1\}$ such that (cf. Exercise Sheet 03, Exercise 04)

$$ (g \circ f \circ g^{-1})(z) = mz. $$

Show that

- $f$ elliptic $\iff a + d \in \mathbb{R}$ and $|a + d| < 2$
  $\iff m = e^{i\varphi} \neq 1$ for some $\varphi \in \mathbb{R}$
- $f$ hyperbolic $\iff a + d \in \mathbb{R}$ and $|a + d| > 2$
  $\iff m = \rho \neq 1$ for some $\rho > 0$.

Hint: Show that $m = \lambda^2$ where $\lambda$ satisfies $\lambda + \frac{1}{\lambda} = a + d$.

(iii) Consider the automorphism of the unit disc

$$ g(z) := \frac{z - q}{qz - 1} $$

for some $q \in \mathbb{C}$, $|q| < 1$. Show that $g$ is elliptic and compute $m$. 
Exercise 2: Möbius transformations with prescribed boundary. (6 pts)

(i) Show that the most general Möbius transformation of the upper half-plane to the unit disc has the form

$$f(z) = e^{i\varphi} \frac{z - a}{z - \bar{a}}$$

where $a \in \mathbb{C}$, $\text{Im}(a) > 0$ and $\varphi \in [0, 2\pi)$.

(ii) Use (i) to show that the most general Möbius transformation of the unit disc to the upper half plane has the form

$$g(z) = \frac{\bar{a}z - ae^{i\varphi}}{z - e^{i\varphi}}.$$

Exercise 3: Two circles under a Möbius transformation. (6 pts)

Let $C_1, C_2 \subset \mathbb{C}$ be two circles. Prove the following statements:

(i) $C_1$ and $C_2$ intersect in two points if and only if there exists a Möbius transformation $f$ such that $f(C_1)$ and $f(C_2)$ are two intersecting lines.

(ii) $C_1$ and $C_2$ touch if and only if there exists a Möbius transformation $f$ such that $f(C_1)$ and $f(C_2)$ are two parallel lines.

Bonus: (2 pts)

Where does the image of the center of a circle under a Möbius transformation lie?

Due Friday, May 15, before the lecture.