Exercise Sheet 7

Exercise 1: Power series expansion. (5 pts)
Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic (i.e. \( f \) an entire function). Let \( A, R > 0 \) and \( m \in \mathbb{N} \) be constants such that
\[
|f(z)| \leq A |z|^m \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > R.
\]
Show that \( f \) is a polynomial of degree at most \( m \).

Exercise 2: Maximum Principle. (5 pts)
Let \( G \subset \mathbb{C} \) be a domain and \( K \subset G \) a compact subset with non-empty interior \( \overset{\circ}{K} \). Show that every non-constant holomorphic function on \( G \) with constant absolute value on the boundary \( \partial K = K \setminus \overset{\circ}{K} \) has a zero in \( K \).

Exercise 3: Reflection Principle. (5 pts)
Show: If \( f \) is an entire function such that \( f(\mathbb{R}) \subset \mathbb{R} \) and \( f(i\mathbb{R}) \subset i\mathbb{R} \), then \( f \) is an odd function, i.e. \( f(-z) = -f(z) \).

Exercise 4: Zeros of holomorphic functions. (5 pts)
Let \( G \) be a domain and \( f : G \to \mathbb{C} \) a holomorphic map. Let \( M := \{ z \in G \mid f(z) \subset \mathbb{R} \} \) and \( z_0 \in M \). Prove the following statements:

(i) If \( f'(z_0) \neq 0 \), then there exists a neighborhood \( V \) of \( z_0 \) in \( G \) such that \( M \cap V \) is the image of a curve \( \gamma : (0, 1) \to G \).

(ii) If \( f'(z_0) = 0 \), i.e. \( n := \text{Ord}(f - f(z_0), z_0) = \min \{ k \in \mathbb{N} \mid f^{(k)}(z_0) \neq 0 \} \geq 2 \), then there exists a neighborhood \( V \) of \( z_0 \) in \( G \) such that \( M \cap V \) is the union of \( n \) curves. Furthermore, the intersection angles of these curves at \( z_0 \) are multiples of \( \frac{2\pi}{n} \).

Bonus: (3 pts)
Calculate and draw the set \( M \) for the following maps: \( \exp, z \mapsto z^3, \sin \).

Due Friday, June 05, before the lecture.