Exercise 1: Residues.  

(i) For $f(z) = \frac{\cot(\pi z)}{z-1}$ compute $\text{res}(f, 0)$ and $\text{res}(f, 1)$.

(ii) For $w \in \mathbb{C} \setminus \mathbb{Z}$ and $f(z) = \frac{\cot(\pi z)}{(z-w)^2}$ show that $\text{res}(f, w) = -\frac{\pi}{\sin^2(\pi w)}$.

Exercise 2: Residues.  

Verify the following integral for $0 < p < 1$:

$$\int_{0}^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} = \frac{2\pi}{1 - p^2}.$$  

Hint: Show that $\int_{0}^{2\pi} \frac{d\theta}{1 - 2p \cos \theta + p^2} = \int_{|z|=1} \frac{dz}{(z-a)(z-a^{-1})}$.

Exercise 3: Rouché’s theorem.  

Let $D := \{ z \in \mathbb{C} \mid |z| < 1 \}$ be the unit circle.

(i) Determine the number of zeros (counting multiplicity) of $f(z) = z^{57} + 36z^{57} + 71z^4 - z + 1$ in $D$.

(ii) Let $a > 1$ be real. Show that the equation $e^{z-a} = z$ has exactly one solution in $D$. Show that this solution is real positive.

(iii) Let $U \subset \mathbb{C}$ be open such that $\overline{D} \subset U$. Let $f : U \to \mathbb{C}$ be holomorphic with $f(\overline{D}) \subset D$. Show that $f$ has exactly one fixed point in $D$. 

Exercise 4: Compact convergence. (6 pts)
Let $G$ be a bounded region and $f : G \to G$ a holomorphic mapping. Let $z_0$ be a fixed point of $f$ with $|f'(z_0)| < 1$. Consider

$$f_n := f \circ \ldots \circ f \text{ n-times}$$

the $n$-th iteration of $f$. Show that $f_n$ converges compactly to $g(z) = z_0$.

Bonus: (4 pts)
Find the partial fraction decompositions of

$$\pi \tan(\pi z).$$

Due Tuesday, July 7, before the lecture.