TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik
Prof. Dr. John Sullivan
Yun Hao

Differential Geometry I - SS 15

http://www3.math.tu-berlin.de/geometrie/Lehre/SS15/DGI/

Homework 11

Problem 1 (2+2+2+2+2 points)

Let $\alpha(s) = \big(r(s), 0, z(s)\big)$ be a unit-speed curve and consider the surface of revolution parametrized by

$$\mathbf{x}(s,\theta) = \big(r(s)\cos\theta, r(s)\sin\theta, z(s)\big).$$

(a) Compute the first fundamental form to verify that these are geodesic parallel coordinates and compute the Christoffel symbols Γ^k_{ij}.

(b) Consider a curve $\gamma(t) = \mathbf{x}\big(s(t), \theta(t)\big)$ on the surface. Using the equations in Problem 2(a) of Homework 10, show that γ is a geodesic if and only if the functions $s(t)$ and $\theta(t)$ satisfy the differential equations

$$\ddot{s} = r(s)r'(s)\dot{\theta}^2, \qquad \ddot{\theta} = \frac{-2r'(s)}{r(s)}\dot{\theta}\dot{s}.$$

(c) Show that all meridians (of constant θ) are geodesics and determine which parallels (of constant s) are geodesics.

(d) Deduce from the equations in part (b) that the scalar products $\langle\dot{\gamma},\dot{\gamma}\rangle$ and $\langle\dot{\gamma},\mathbf{x}_\theta\rangle$ are constant along a geodesic. The first of these of course just says that the geodesic is parametrized at constant speed $|\dot{\gamma}|$. Reinterpret the second as *Clairaut's relation*: if ψ denotes the angle between the γ and the parallels, then $r\cos\psi$ is constant along a geodesic.

(e) Explicitly determine all geodesics on the unit sphere and on the unit cylinder

$$\mathrm{x}(s,\theta) = (\cos\theta, \sin\theta, s).$$

Problem 2 (3 points)

Let $\alpha(t)$ be a curve on a surface M and let κ_g denote its geodesic curvature at $p := \alpha(0)$. Let $\tilde{\alpha}$ be the orthogonal projection of α onto T_pM. Show that the curvature κ of $\tilde{\alpha}$ at p is $\kappa = |\kappa_g|$.

Problem 3 (3 points)

Let α be a curve in a surface M and let X and Y be two vector fields tangent to M along α. Show that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\langle X(t), Y(t)\rangle = \Big\langle \frac{\nabla}{\mathrm{d}\,t}X, Y\Big\rangle + \Big\langle X, \frac{\nabla}{\mathrm{d}\,t}Y\Big\rangle$$

Homework due at the start of lecture on Monday 6 July.