TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. John Sullivan
Hana Kouřimská

Geometry II - SS 16
Discrete Differential Geometry

http://www.math.tu-berlin.de/geometrie/Lehre/SS16/GeometryII/

Exercise Sheet 4

Exercise 1: Hyperbolic geodesics in the halfplane model. (6 pts)

For regular curves $\gamma : [s_0, s_1] \to H^2_+ = \{\binom{x_1}{x_2} \ : \ x_2 > 0\}$ consider the hyperbolic length:

$$L_H(\gamma) = \int_{s_0}^{s_1} \frac{1}{\langle \gamma, e_2 \rangle} \sqrt{\langle \gamma', \gamma' \rangle} ds.$$

Show that a curve γ is an extremum of L_H for all variations γ_t of γ with fixed endpoints, if and only if

$$\langle \gamma, e_2 \rangle \kappa N - \langle T, e_2 \rangle T + e_2 = 0.$$

Deduce that $\kappa' = 0$. (Hint: Scalar multiply the previous equation with N.)
Show further:

- If $\kappa = 0$, then γ lies on a vertical line.
- If $\kappa \neq 0$, then γ is an arc of a circle with center on the e_1-axis.

(note about using the scalar curvature)

Exercise 2: The Brachistochrone problem. (7 pts + 4 extra pts)

The aim of this exercise is to prove the Brachistochrone problem:
"Find a curve in a vertical plane connecting $A = (0, 0)$ with $B = (x_1, y_1), y_1 < 0$ so that a particle sliding down this curve under gravity from A arrives at B in the shortest time."

(a) Prove that if the Lagrangian F depends explicitly only on q and q', then the Euler-Lagrange differential equation becomes

$$F - q' F_{q'} = C$$

where C is a constant.

(b) Parameterize an admissable curve

$$\gamma : [0, x_1] \to \mathbb{R}^2, x \mapsto \gamma(x) \text{ with } \gamma(0) = A, \gamma(x_1) = B$$

as a graph of a function.

(c) Construct the functional keeping in mind the dependence of the velocity of the particle on gravity:

$$\frac{ds}{dt} = \sqrt{2gy},$$

where g is the gravity constant, y is an admissable function, dt is the time element and ds is the length element.

(d) Calculate the Euler-Lagrange equation.

(e) (**additional exercise**) Solve the Euler-Lagrange equation.

PLEASE TURN!!!

Exercise 3: Surfaces of revolution. (3 pts)

Find the curve

$$\gamma : [0, 1] \to \mathbb{R}^3, x \mapsto (x, y(x), 0), \gamma(0) = (0, a, 0), \gamma(1) = (1, b, 0), a, b > 0$$

such that the surface of revolution created by rotating this curve about the x-axis has the least area, i.e. find a minimizer of the following functional:

$$S[y] = 2\pi \int_0^1 y\sqrt{1 + y'^2}dx.$$

It is enough to express $y(x)$ using the constants of integration. You do not need to determine the constants of integration in dependence of a and b.
Hint: Use the ansatz $y' = \sinh t$.

Due: Lecture on May 30, 2016.