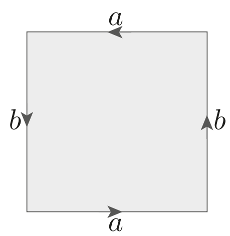
Mathematical Visualization

Assignment 2 - The real projective plane[1]

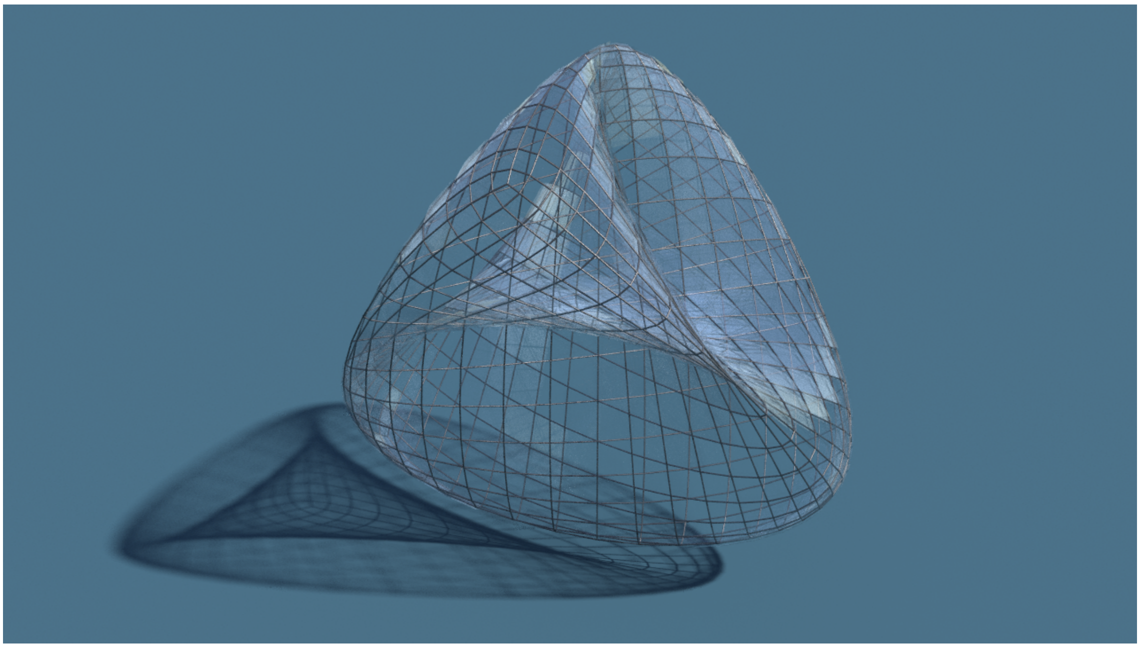
The *real projective plane* $\mathbb{R}\mathrm{P}^2$ is obtained by identifying the antipodal points of the a 2-sphere $\mathbb{S}^2$, i.e. $\mathbb{R}\mathrm{P}^2 = \mathbb{S}^2/_\sim$ with equivalence relation given by

$$x \sim y \Leftrightarrow y = \pm x.$$

The real projective plane is a sphere with *cross-cap* - the fundamental polygon looks as follows:



The roman surface: It is known that there exists no embedding of $\mathbb{R}\mathrm{P}^2$ into Euclidean 3-space. Though it can be mapped to $\mathbb{R}^3$ with self-intersection. An example is Steiner's *roman surface.*



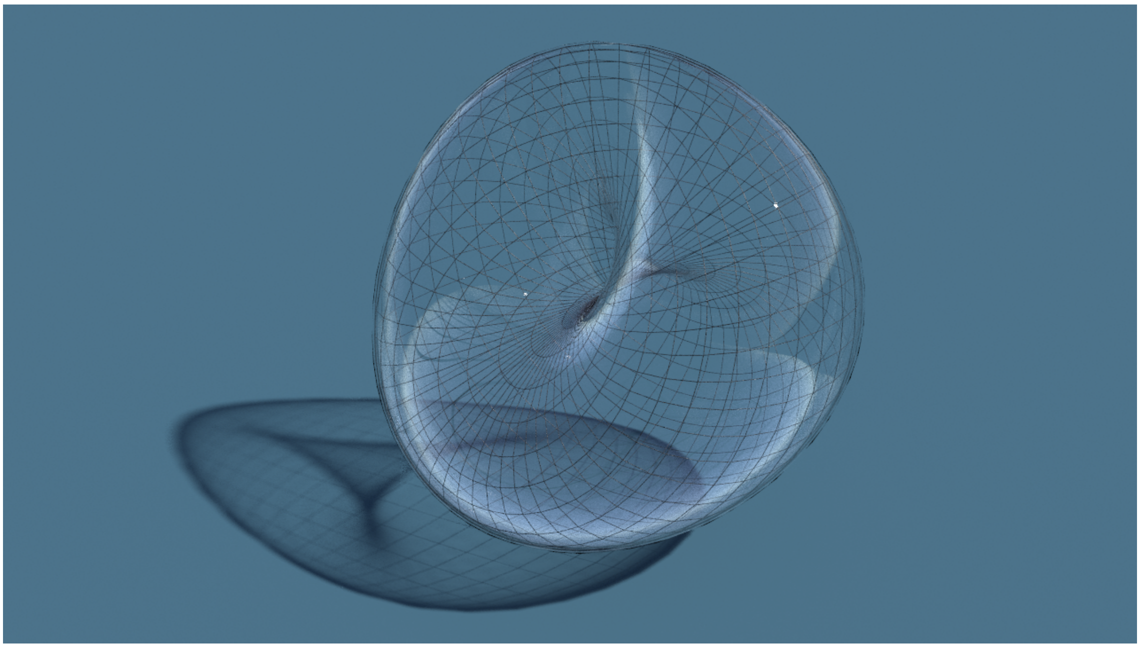
The corresponding map is given as follows:

$$f\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \quad [(x,y,z)] \mapsto \big(yz, zx, xy\big).$$

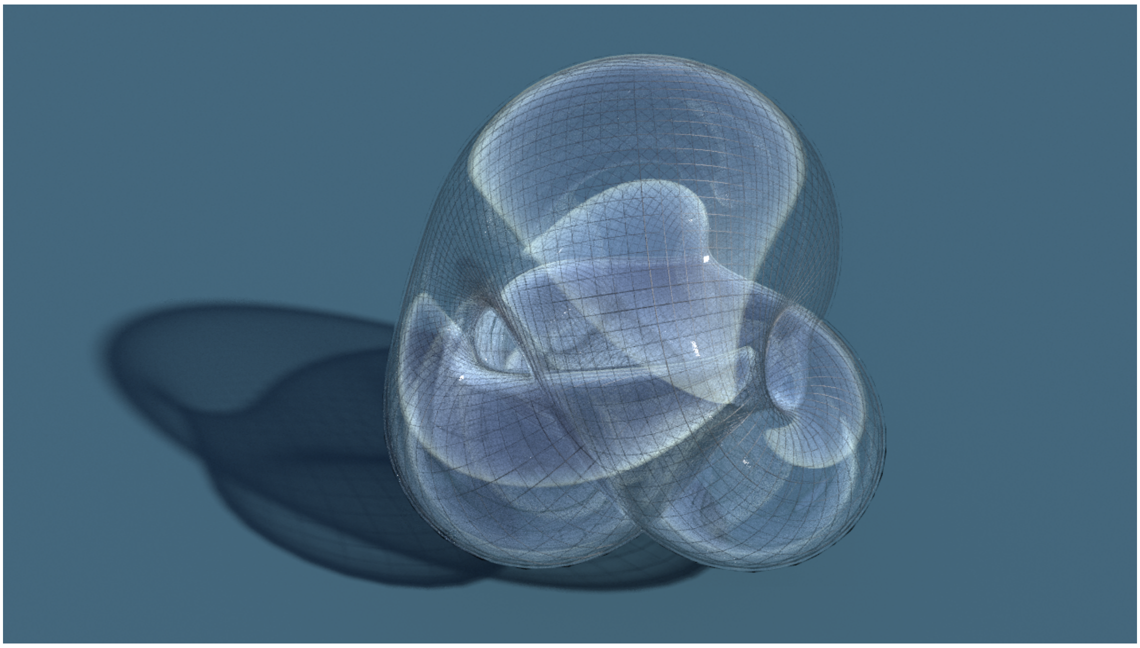
[1] due 14.5.2017

Cross-cap: Actually the roman surface is a projection of the *Veronese surface* into Euclidean 3-space and as such a special case of a *general construction method for non-orientable surfaces.* Choosing a different projection we also obtain the cross cap from it:

$$f\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \quad [(x, y, z)] \mapsto \big(yz, zx, \tfrac{1}{2}(z^2 - x^2)\big).$$



Boy's surface: The maps above have singularities and it was not clear that an immersion exists until 1901. At that time Boy discovered an immersion which nowadays is known as *Boy's surface.* It can be realized in different ways.



We will use the realization above due to Kusner and Bryant. It is especially beautiful in the sense that it is a minimizer of the *Willmore energy* - a so called Willmore surface.

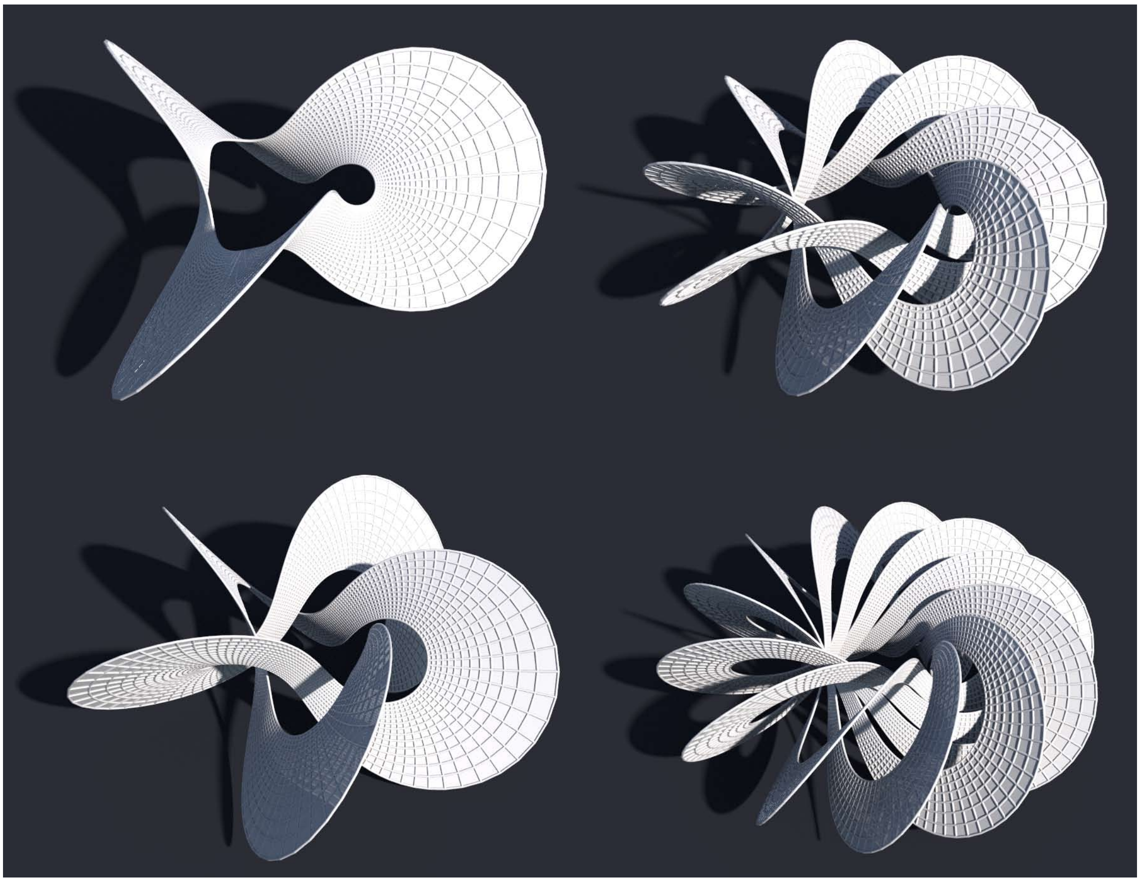
It is the inversion of a complete minimal surface with 3 ends and as a consequence rational, i.e. the immersion can be written down explicitly with rational functions. Actually it is *known* that all Willmore surfaces parametrized by spheres are obtained in this way.

This week your task is to visualize Boy's surface. Most things work as in last week's assignment: Everything will be done on $\mathbb{S}^2$, which is an orientable double-cover of $\mathbb{RP}^2$.

Kusner surfaces: *Kusner* gave an explicit formula for complete minimal surfaces with ends: Let $p \in \mathbb{N}$ and $r = 2\sqrt{2p-1}/(p-1)$. Let $z\colon \mathbb{S}^2 \to \mathbb{C}$ be the stereographic projection. Then the immersion $\mathbb{S}^2 \to \mathbb{R}^3$ is defined to be the real part of

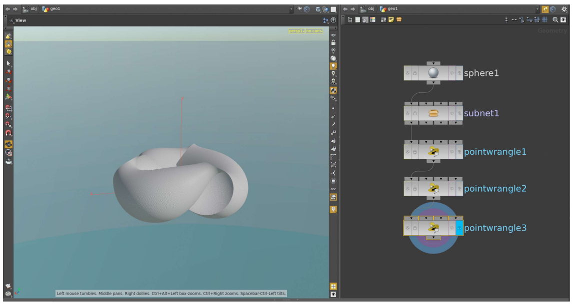
$$\Phi_p = \frac{i}{z^{2p} + rz^p - 1}\Big(z^{2p-1} - z, -i(z^{2p-1} + z), \tfrac{p-1}{p}(z^{2p} + 1)\Big).$$

The integer p controls the symmetry of the immersed surface. Below the surfaces for $p = 3, 4, 5$ and 6. Here we have cut out disks around the poles of $\mathbb{S}^2$ for visualization purposes.



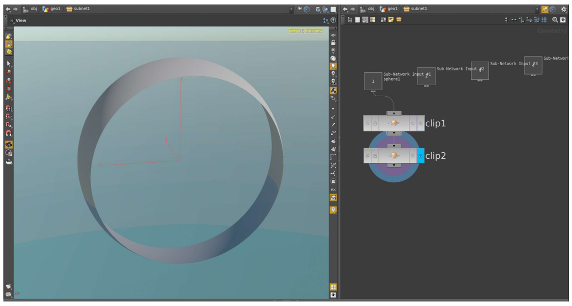
Finally we obtain Boy's surface by a Möbius transformation of the Kusner surface with $p = 3$. The transformation is an inversion in the unit sphere with center $(0, 0, h)$, $h \neq 0$.

Implementation: The implementation is straight forward. The network could look as follows:

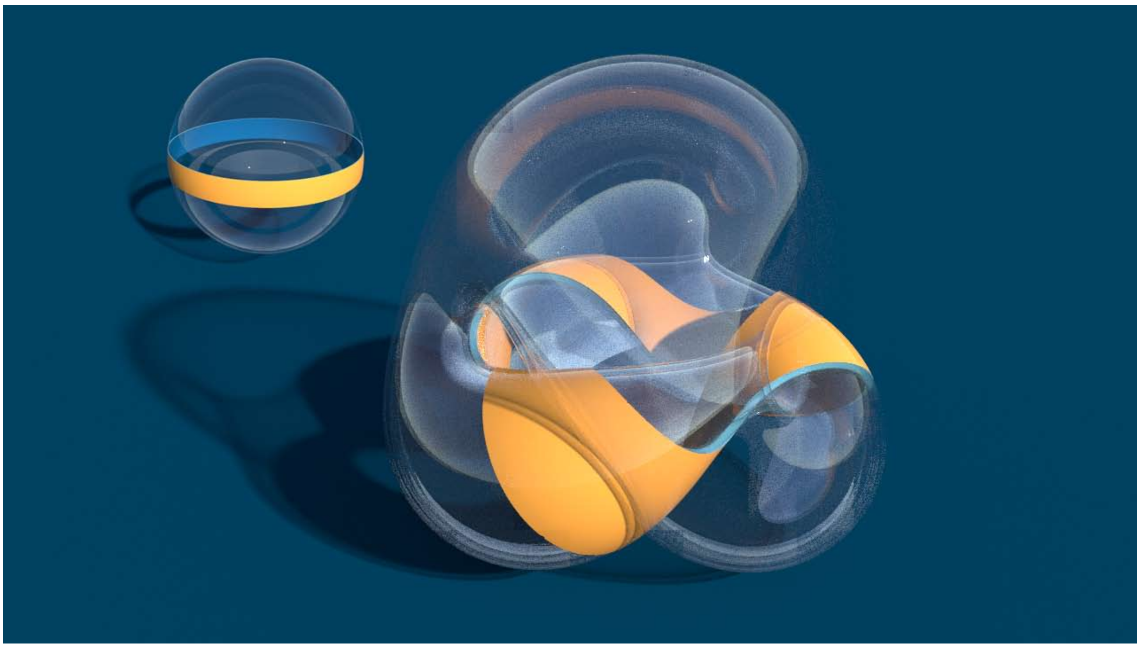


We start with a sphere, apply the stereographic projection (`poinwrangle1`), Kusner's formula (`pointwrangle2`) and finally the inversion in the sphere (`pointwrangle3`).

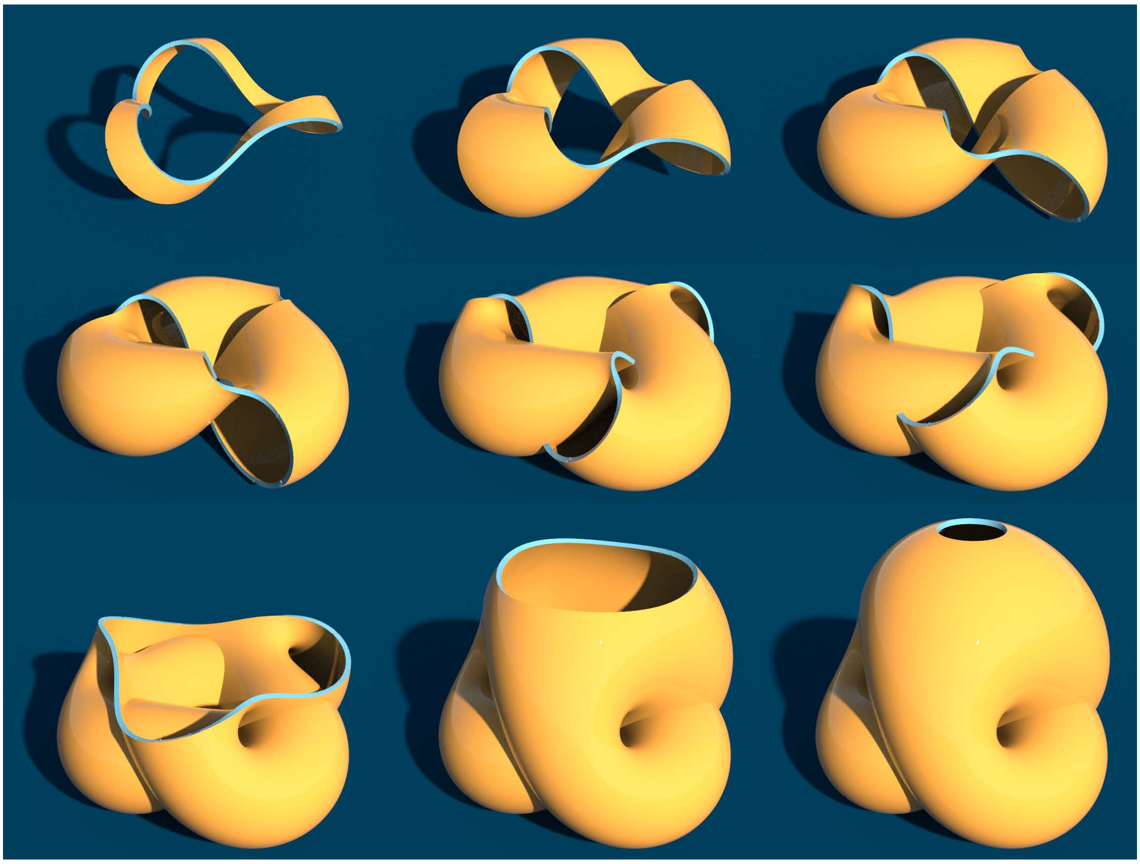
The node `subnet1` contains a small network that cuts off a cap from the north pole and a cap from the south pole of thesphere. This can be done by two *clip nodes*.



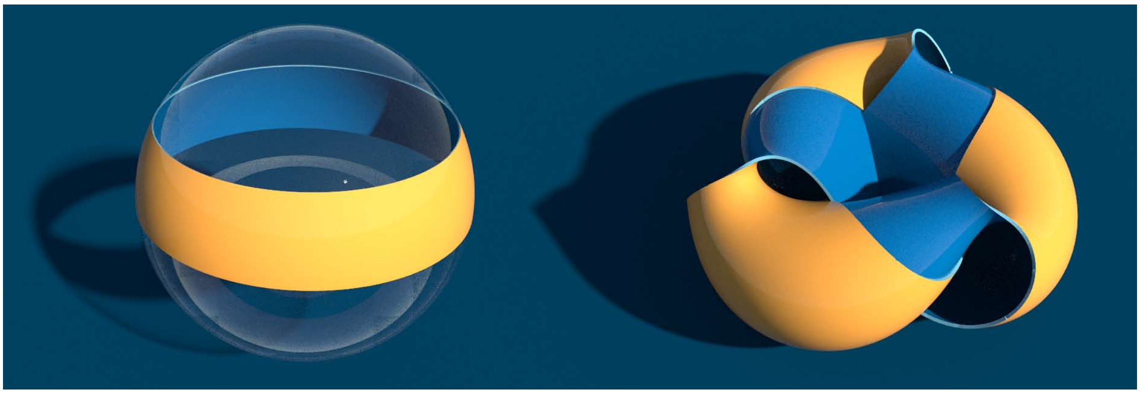
Cutting this holes symmetrically corresponds to cutting a disk from $\mathbb{R}\mathrm{P}^2$. Thus the cylinder above is mapped to a Möbius strip on Boy's surface.



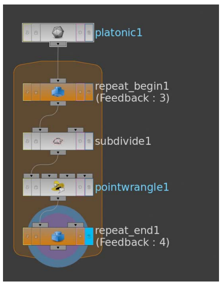
Though the surface is still quite hard to understand from the picture. One way to understand it better is to use a parameter for the width of the strip. Here a sequence for growing width.



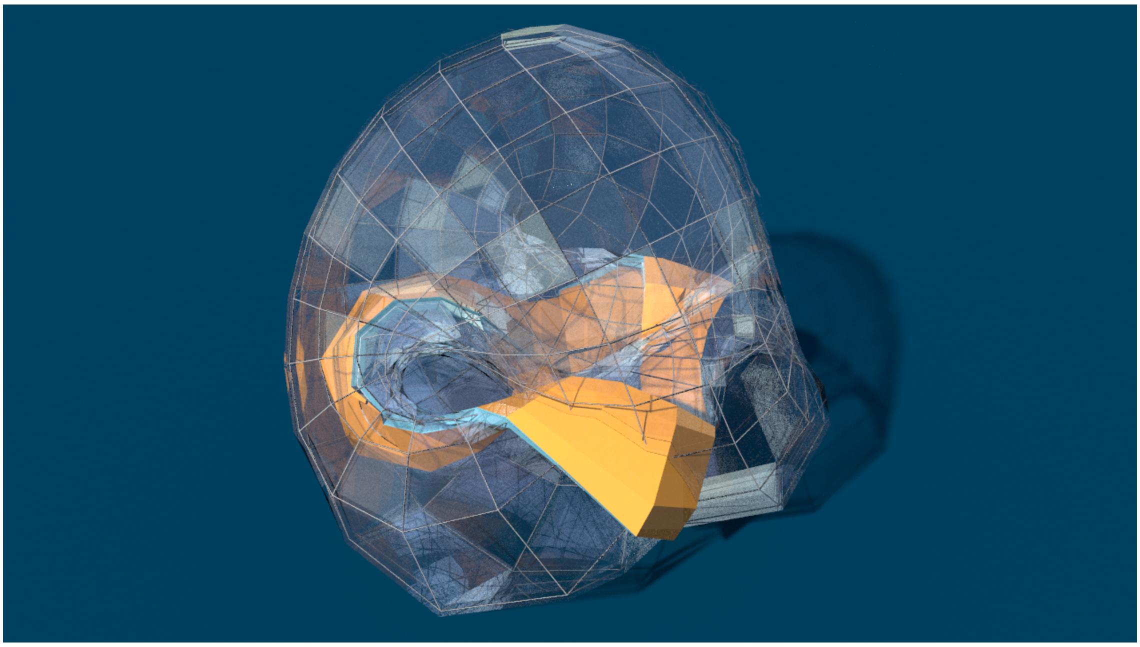
If we move the strip up and destroy the symmetry the two sheets of the covering become visible.



Note that a cell decomposition of $\mathbb{R}\mathrm{P}^2$ is basically a cell decomposition of $\mathbb{S}^2$ which is mapped by the antipodal map to itself. One way to obtain such a cell decomposition is by subdivision. E.g. we can start with a tetrahedron, then apply on subdivision step using *Cutmull-Clark subdivision* before we project back to the sphere. To obtain finer meshes one can *iterate this*.



Below a quite coarse parametrization of Boy's surface.



Homework 2: Wire up a network to visualize Boy's surface. In detail this means:

- Write a subnet that produces a cell decomposition of $\mathbb{R}\mathrm{P}^2$.
- Write a subnet that removes disks around north and south pole to obtain a strip.
- Implement Kusner's formula and postcompose with the sphere inversion to obtain Boy's surface.

Build in parameters to control the resolution of the cell-decomposition, the integer p and the width of the strip.