Mathematical Visualization

Assignment 6 - Hopf tori[1]

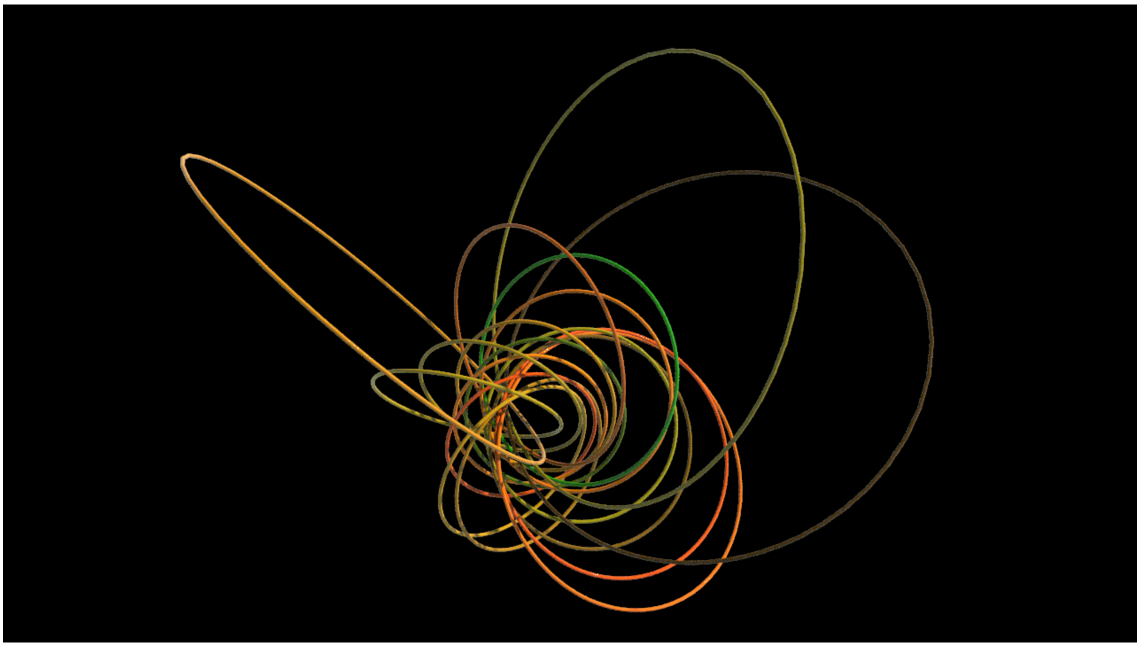
Consider the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{H}$ and identify Euclidean 3-space with the imaginary quaternions, $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$. The map $\pi\colon \mathbb{S}^3 \to \mathbb{S}^2$ given by

$$q \mapsto q\mathbf{i}\bar{q}$$

is called *Hopf fibration*. One can check that the preimage $\pi^{-1}(\{p\})$ of any point $p \in \mathbb{S}^2$ is a great circle in $\mathbb{S}^3$ - a so called *Hopf circle*. The 3-sphere is then the disjoint union of Hopf circles:

$$\mathbb{S}^3 = \dot{\bigcup_{p\in\mathbb{S}^2}} \pi^{-1}(\{p\}).$$

We can then use the stereographic projection to visualize the Hopf circles in Euclidean 3-space. The Hopf circles of a random set of points on $\mathbb{S}^2$ are shown in the picture below.

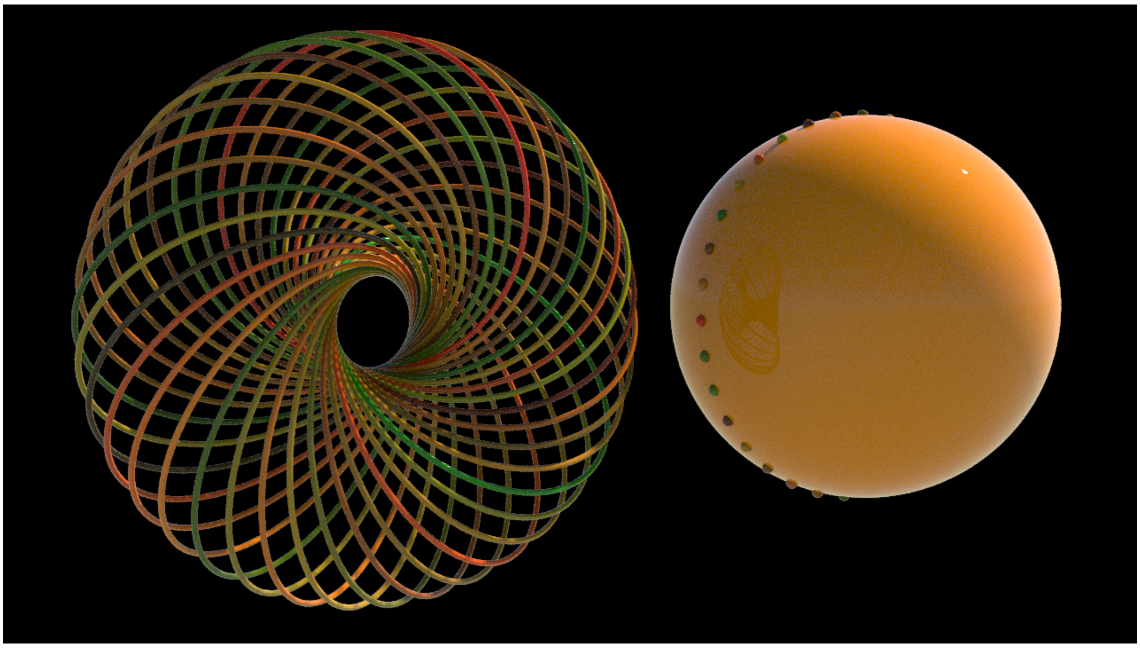


Here we just sampled the smooth parametrization to draw the Hopf circles: If $q \in \mathbb{S}^3$, then the Hopf circle through q can be parametrized by

$$t \mapsto qe^{\mathbf{i}t}.$$

Similarly, the points of a discrete spherical curve $\gamma = (\gamma_1, \ldots, \gamma_n)$ determine n Hopf circles.

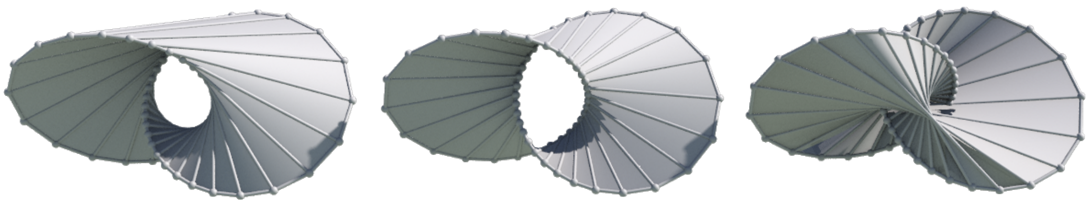
[1]due 2.7.2017



If we connect these circles they form a discrete cylinder or, if γ is closed, a torus: Given a *lift* of γ, i.e. a discrete curve $\psi = (\psi_1, \ldots, \psi_n)$ in $\mathbb{S}^3$ such that $\pi \circ \psi = \gamma$, we obtain a discrete parametrization

$$f_{ij} = \psi_i e^{2\pi \mathbf{i} \frac{j}{m}}.$$

The choice of the lift ψ has a strong effect on the quality of the parametrization, as demonstrated for just two circles in the pictures below.



A particularly nice parametrization can be achieved if we use a horizontal lift: Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a discrete spherical curve such that $\gamma_{i+1} \neq \pm\gamma_i$ for all i. We define

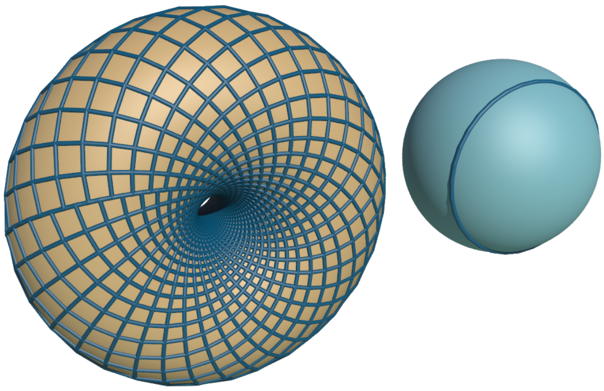
$$r_i = \cos\tfrac{\alpha}{2} + \sin\tfrac{\alpha}{2}\tfrac{\gamma_i \times \gamma_{i+1}}{|\gamma_i \times \gamma_{i+1}|},$$

where $\alpha = \arccos\langle\gamma_i, \gamma_{i+1}\rangle$. Then left-multiplication with r_i yields a map $\pi^{-1}(\{\gamma_i\}) \mapsto \pi^{-1}(\{\gamma_{i+1}\})$ and thus an identification of the Hopf circles over the points of γ. A lift $\psi = (\psi_1, \ldots, \psi_n)$ of γ is then called *horizontal*, if

$$\psi_{i+1} = r_i \psi_i, \quad \textit{for all } i.$$

The definition of r_i can be derived from horizontal lifts along great circle arcs. Details can be found in a *post* from the previous semester.

A parametrization of a Hopf torus corresponding to a circle on $\mathbb{S}^2$ is shown in the picture below. As you can see the parametrization fails to close up. This comes from the fact that the horizontal lift ψ of a periodic curve γ does not need to be periodic.



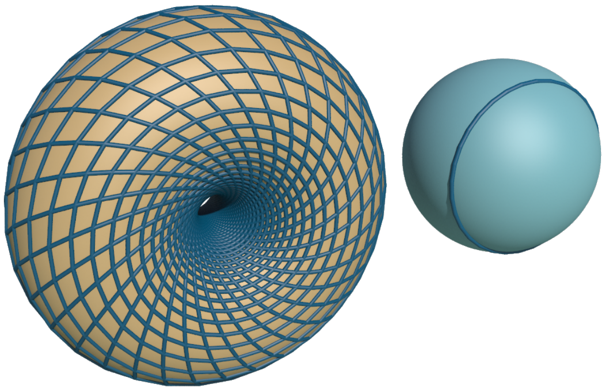
Given a closed spherical curve $\gamma = (\gamma_1, \dots, \gamma_n)$ and the corresponding discrete transports r_i as described above. Then the map $q \mapsto r_n \cdots r_1 \, q$ maps the Hopf circle $\pi^{-1}(\{\gamma_1\})$ to itself. Thus there is $\omega \in \mathbb{R}$ such that

$$r_n \cdots r_1 \psi_1 = \psi_n e^{-\mathbf{i}\omega}.$$

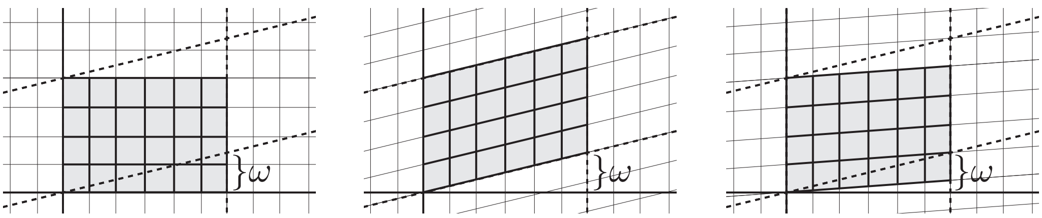
This angle defect ω we already know in the context of parallel frames. It is related to the geometry of γ. If we insist on a parametrization which is periodic in j we can distribute the angle defect equally to all edges: If ψ is a horizontal lift, we define

$$\phi_j = \psi_j e^{\mathbf{i}\frac{\omega}{n} j}.$$

Then ϕ is a closed lift of γ. Below a picture of the corresponding parametrization.

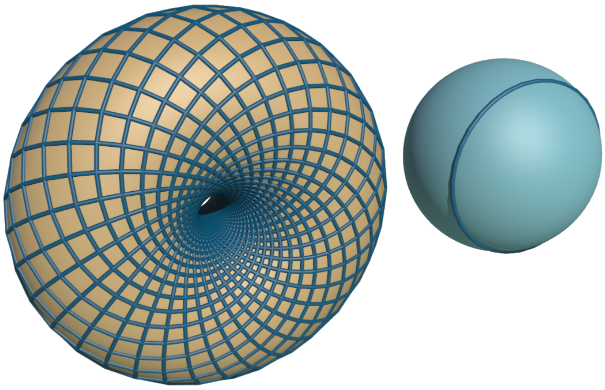


Actually we can do better - it would be enough if the parameter lines close up. Thus we can also use a smaller phase which just maps the points onto each other.

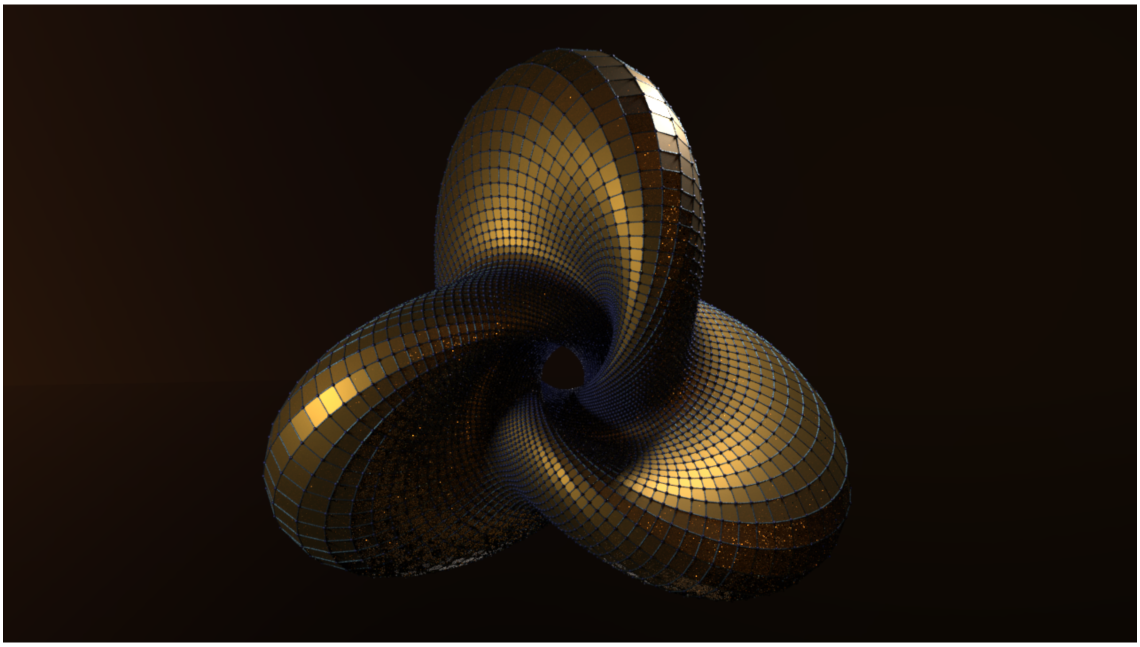


Here the horizontal lines correspond to horizontal lifts and the dashed thick lines mark a fundamental domain of the torus - they correspond to the smooth counterpart of the lift ϕ defined above.

Below a picture of an example of a parametrization with parameter lines closing up.



This explained we can produce more interesting tori. Below a torus which comes which corresponds to the intersection curve of a cubical cone with $\mathbb{S}^2$ also described in the *DDG2016 blog*.



Exercise 6: Wire up a network that generates for a given discrete spherical curve a discrete parametriation with closed parameter lines as described above.