Berlin Mathematical School

TU Berlin
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http://www3.math.tu-berlin.de/geometrie/Lehre/WS12/MPI/

WS 2012/2013

Mathematical Physics I
Exercises 5

Problem 1:
Consider the following system of differential equations:

$$\dot{x} = -\frac{x}{2} - y^2, \tag{1}$$
$$\dot{y} = -\frac{y}{4} - x^2. \tag{2}$$

1. Linearise the system around the fixed point $(0,0)$.
2. Show that $E(x,y) := x^2+y^2$ is a Liapunov-function for the linear system. Investigate the stability of the fixed point of the linearized system.
3. Show that the Liapunov-function defined in 2) is a Liapunov-function for the non-linear system, too.

(6 points)

Problem 2:
Show that $(0,0)$ is a fixed point of the following two systems of differential equations, which is in both cases linearly stable, but unstable in case a) and asymptotically stable in case b).

a) $\dot{x} = -y + 2x^5$, $\dot{y} = 2x + y^3$, b) $\dot{x} = -y - 2x^5$, $\dot{y} = 2x - y^3$.

Hint: Construct a suitable Liapunov-function. (8 points)

Problem 3: Van der Pol-Oscillator

Consider the differential equation of the Van der Pol-Oscillator:

$$\ddot{x} - \epsilon\left(1 - x^2\right)\dot{x} + x = 0 \ . \tag{3}$$

This is the differential equation for the electric current $x(t)$ in an LRC-resonant circuit with a non-linear amplification (a triode).
Consider the dynamical system

$$\begin{aligned} \Phi : \qquad & \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \\ & t, (x_0, y_0) \mapsto (x, y) \ , \end{aligned} \tag{4}$$

with x und y solutions of the initial value problem (3), $x(0) = x_0$, $y(0) = y_0$.

1. Let $\epsilon < 0$. Show that the fixed point $(0, 0)$ is asymptotically stable in this case by constructing a suitable Liapunov-function.
   Hint: A canditate for the Lyapunov function is given by the integral of motion of the differential equation with $\epsilon = 0$.

2. Prove that the mapping $\epsilon \to -\epsilon$ is equivalent to the time reversal $t \to -t$. Using this result, what can you say about the stability of $(0, 0)$ if $\epsilon > 0$?

(6 points)

The next office hour of Christina Papenfuss is Mo 19.11., 15.30 - 16.30 instead of Tuesday 20.11.