Berlin Mathematical School

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http://www3.math.tu-berlin.de/geometrie/Lehre/WS12/MPI/

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Mathematical Physics I
Exercises 6

Problem 1:
Consider the dynamical system

$$\begin{aligned} \Phi : \quad & \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \\ & t, (x_0, y_0) \mapsto (x, y) \ , \end{aligned} \tag{1}$$

with x and y solutions of the initial value problem

$$\dot{x} = ax - y + kx\left(x^2 + y^2\right) \tag{2}$$

$$\dot{y} = x - ay + ky\left(x^2 + y^2\right) \ , \ a^2 < 1 \tag{3}$$

$$x(0) = x_0 \ , \ y(0) = y_0$$

Prove that this system has an asymptotically stable fixed point in the case $k < 0$:

1. Prove that for any fixed point (x_F, y_F)

   $$x_F^2 - 2ax_Fy_F + y_F^2 = 0 \ . \tag{4}$$

   Hint: Make use of the differential equations (2) and (3)

2. Show that the curves in the $x-y$-plane with $v(x, y) := x^2 - 2axy + y^2 = C$ ($C \in \mathbb{R}$) are ellipses around the origin. For fixed C determine the values of the two major axes. Which is the angle between the major axis and the x- or y-axis, respectively? Sketch the ellipses.

3. Conclude that $(x_F, y_F) = (0, 0)$ is the only fixed point.

4. The normal vector to the ellipses (pointing outward) ∇v has the components $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$. Calculate ∇v.

5. ψ denotes the angle between the tangent to the orbit in a point and normal vector ∇v in this point. Calculate $\cos\psi$.

6. Show that for $k < 0$ it follows $\cos\psi \leq 0$. Is $\psi = 0$ possible?

7. Sketch an orbit with $\cos\psi \leq 0$ in each point.

8. In the case $k < 0$, conclude that the fixed point $(0,0)$ is asymptotically stable.
   Hint: After entering one of the ellipses, is it possible, that the orbit leaves the ellipse again?

9. Investigate the same arguments in the case $k > 0$.

(9 points)

Problem 2: LotkaVolterra equations

The following differential equations describe the dynamics of biological systems in which two species interact, one a predator and one its prey. x is the number of prey (for example, rabbits) and y is the number of some predator (for example, foxes) $(x, y \geq 0)$.

$$\begin{aligned} \dot{x} &= (\alpha - \beta y)\, x \\ \dot{y} &= (\delta x - \gamma)\, y \;, \quad \alpha, \beta, \gamma, \delta > 0 \;. \end{aligned} \tag{5}$$

1. Find the two fixed points and linearize the differential equations around the fixed points.

2. For which one of the fixed points is the theorem of Grobman-Hartmann applicable? Decide about the stability of the fixed points.

(5 points)

Problem 3:

Consider the following linear systems of ordinary differential equations:

$$\begin{aligned} \dot{x} &= -2x \\ \dot{y} &= -2y \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{x} &= -2x + y \\ \dot{y} &= -2y - x \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{x} &= -2x \\ \dot{y} &= -2y + x \end{aligned} \tag{8}$$

1. Sketch all three phase portraits.

2. Prove that the above systems are all topologically conjugate. Find explicitly the topological mapping.
   Hint: In one case, it is convenient to introduce polar coordinates.

(6 points)