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http://www3.math.tu-berlin.de/geometrie/Lehre/WS14/DiffGeoII WS 14

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 2

(Smooth manifolds)

due 6.11.2014

Exercise 1 **3 points**

Let M be a topological manifold. Show that M has an *exhaustion by compact sets*, i.e. there exists a sequence $K_1 \subset K_2 \subset K_2 \subset \cdots \subset \mathrm{M}$ of compact sets such that $K_i \subset \mathring{K}_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = \mathrm{M}$.

Exercise 2 **4 points**

Let M and N be smooth manifolds of dimension m and n, respectively. The topological product $\mathrm{M} \times \mathrm{N}$ is a topological manifold of dimension $m+n$.

Let $\{(U_i, \varphi_i)\}_{i\in I}$ a smooth atlas for M and $\{(V_j, \psi_i)\}_{j\in J}$ a smooth atlas for N. Show that $\{(U_i \times V_j, \varphi_i \times \psi_j)\}_{(i,j)\in I\times J}$ is a smooth atlas for $\mathrm{M} \times \mathrm{N}$.
Here $\varphi_i \times \psi_j \colon U_i \times V_j \to \varphi_i(U_i) \times \psi_j(V_j)$ with $(\varphi_i \times \psi_j)(x, y) = (\varphi_i(x), \psi_j(y))$.

Exercise 3 **4 points**

Let $n \geq 0$ and let $J = \{0, \ldots, n\}$. We define $\left(U_j^{\pm}, \varphi_j^{\pm}\right)_{j\in J}$ by

$$U_j^{\pm} := \left\{x = (x_0, \ldots, x_n) \in \mathbb{S}^n \mid \pm x_j > 0\right\},$$
$$\varphi_j^{\pm} \colon U_j^{\pm} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Show that $\left(U_j^{\pm}, \varphi_j^{\pm}\right)_{j\in J}$ is a $\mathscr{C}^\infty$-atlas for $\mathbb{S}^n$.

Exercise 4 **4 points**

$\mathrm{M} := \{P \in \mathrm{Mat}(3\times 3, \mathbb{R}) \mid P^2 = P,\ P^* = P,\ \mathrm{tr}(P) = 1\} \subset \mathrm{Sym}(3) \cong \mathbb{R}^6$.
Show that M is a submanifold diffeomorphic to $\mathbb{R}\mathrm{P}^2$.