Midterm Exam – Topology

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You have **80 minutes** to complete this exam. There are three exercises worth 10 points each. The exam is passed successfully with **15 points** or more. You may write your solutions in either German or English. Always show that the prerequisites of theorems you use are satisfied.
Exercise 1 10 points

Let

- \( X = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\} \), that is, \( X \) is \( \mathbb{R}^3 \) with a line removed,
- let \( Y = \mathbb{R}^2 \setminus \{(-1, 0), (1, 0)\} \),
- and let \( Z = S^1 \subseteq \mathbb{R}^2 \) be the unit circle.

Decide which pairs of spaces \( X, Y, Z \) are homotopy equivalent, that is, whether \( X \simeq Y \), \( Y \simeq Z \), and \( X \simeq Z \). Explain your answer.

Exercise 2 10 points

Let \( u_0 \in S^1 \) be fixed. Construct the space \( X \) by gluing a disk into the torus \( S^1 \times S^1 \) along the loop \( \{u_0\} \times S^1 \) via the map \( f: S^1 \to S^1 \times S^1, x \mapsto (u_0, x) \). For some \( x_0 \in X \) compute \( \pi_1(X, x_0) \).

Exercise 3 10 points

Let \( X = \mathbb{R}P^2 \vee \mathbb{R}P^2 \).

(a) Argue that \( \pi_1(X) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2 \).

(b) Let \( Y = \mathbb{R}P^2 \vee S^2 \vee \mathbb{R}P^2 \), where a point of one copy of \( \mathbb{R}P^2 \) is identified with the north pole of \( S^2 \) and a point of the other copy of \( \mathbb{R}P^2 \) is identified with the south pole. Construct a covering map \( p: Y \to X \) and determine the subgroup \( p_*(\pi_1(Y, y_0)) \) of \( \pi_1(X, p(y_0)) \) for a basepoint \( y_0 \in Y \) of your choice.

(c) Construct the universal covering of \( X \) geometrically (for instance, as a map from a CW-complex to \( X \)).