Practice Test – Topology

The midterm exam will take place on December 17, 2:15 p.m. in MA 313. Please be there a couple of minutes early. You may bring an A4 sheet of handwritten notes. You will have 80 minutes to complete the exam.

Exercise 1 10 points

Let

- \( X = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\} \), that is, \( X \) is \( \mathbb{R}^3 \) with a line and a circle around this line removed,
- \( Y = \mathbb{R}^2/\sim \), where \((x, y) \sim (x', y')\) if and only if \( y = y' \) and \( x - x' \in \mathbb{Z} \), and
- let \( Z = S^1 \times S^1 \) be the torus.

Decide which pairs of spaces \( X, Y, Z \) are homotopy equivalent, that is, whether \( X \simeq Y \), \( Y \simeq Z \), and \( X \simeq Z \).

**Solution.** We first show that \( Y \simeq S^1 \). Denote by \( p : \mathbb{R}^2 \to Y \) the canonical quotient map, and let \( H : Y \times [0, 1] \to Y, ([x, y], t) \mapsto p(x, (1-t)y) \). This map is well-defined and a deformation retraction from \( Y \) onto \( p(\mathbb{R} \times \{0\}) \cong S^1 \). Thus \( Y \) and \( Z \) are not homotopy equivalent since \( \pi_1(Y) \cong \mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z} \cong \pi_1(Z) \).

We will now show that \( X \simeq Z \). By transitivity of homotopy equivalence it also follows that \( X \not\simeq Y \).

By applying the inverse stereographic projection, \( X \) is homeomorphic to \( S^3 \) with the standard Hopf link \( L = \{(x_1, \ldots, x_4) \in S^3 \mid x_3 = x_4 = 0\} \cup \{(x_1, \ldots, x_4) \in S^3 \mid x_1 = x_2 = 0\} \) removed. It was shown in the lecture that \( S^3 \setminus L \cong \mathbb{R} \times S^1 \setminus S^1 \simeq S^1 \times S^1 \).

Exercise 2 10 points

Let \( K = [(0, 1] \times [0, 1])/\sim \), where the equivalence relation \( \sim \) is generated by \((0, y) \sim (1, y)\) and \((x, 0) \sim (1-x, 1)\), that is, \( K \) is the Klein bottle. Denote by \( p : [0, 1] \times [0, 1] \to K \) the corresponding quotient map.

Let \( U = p((\frac{1}{4}, \frac{3}{4}) \times [0, 1]) \) and \( V = p([0, \frac{1}{4}] \times [0, 1]) \cup p((\frac{3}{4}, 1] \times [0, 1]) \). Check that the prerequisites of van Kampen’s theorem are satisfied, and use it to compute the fundamental group of \( K \) from those of \( U \) and \( V \).

**Solution.** The set \((\frac{1}{4}, \frac{3}{4})\times [0, 1] = ((\frac{1}{4}, \frac{3}{4}) \times \mathbb{R}) \cap ([0, 1] \times [0, 1])\) is open in \([0, 1] \times [0, 1]\) by definition of subspace topology. By definition of quotient topology \( U = p((\frac{1}{4}, \frac{3}{4}) \times [0, 1]) \) is also open. The same reasoning applies to \( V \) since \([0, \frac{1}{4}] \times [0, 1] = (-1, \frac{1}{4}) \times \mathbb{R}) \cap ([0, 1] \times [0, 1])\) and \((\frac{3}{4}, 1] \times [0, 1] = ((\frac{3}{4}, 2) \times \mathbb{R}) \cap ([0, 1] \times [0, 1])\).

The intersection \( U \cap V = p((\frac{1}{4}, \frac{1}{2}) \times [0, 1]) \cup p((\frac{3}{4}, \frac{1}{2}) \times [0, 1]) \) has at most two path-components as continuous image of two path-connected sets. In fact, \( U \cap V \) has only one path-component. To see this it suffices to find a path from a point in \( p((\frac{1}{4}, \frac{1}{2}) \times [0, 1]) \) to a point in \( p((\frac{3}{4}, \frac{1}{2}) \times [0, 1]) \). Let \( \gamma_1 : [0, 1] \to [0, 1] \times [0, 1], t \mapsto ((\frac{1}{2}, t), (1-t)\frac{1}{2}) \) and \( \gamma_2 : [0, 1] \to [0, 1] \times [0, 1], t \mapsto (\frac{7}{10}, t\frac{1}{2}) \). Then \((p \circ \gamma_1) \cdot (p \circ \gamma_2)\) is such a continuous path.

The set \( U \) deformation retracts to \( p((\frac{1}{2}) \times [0, 1]) \cong S^1 \), and \( V \) deformation retracts to \( p([0) \times [0, 1]) \cong S^1 \). The intersection deformation retracts to \( p((\{\frac{1}{2}\} \times [0, 1]) \cup (\{\frac{7}{10}\} \times [0, 1]) \) \cong S^1. Thus \( \pi_1(U) \cong \langle a \rangle \), \( \pi_1(V) \cong \langle b \rangle \), and \( \pi_1(U \cap V) \cong \langle c \rangle \). Denote by \( \iota_U : U \cap V \to U \) and \( \iota_V : U \cap V \to V \) the inclusions. Then \( (\iota_U)_*(c) = a^2 \) and \( (\iota_V)_*(c) = b^2 \).
Exercise 3 10 points

Let $X$ be the space obtained from an $n$-gon (that is a 2-cell bounded by $n$ 1-cells) by identifying all edges with the same orientation around the $n$-gon.

(a) Argue that $\pi_1(X, x_0) \cong \mathbb{Z}/n$.

(b) Let $X$ be the space that is obtained from $n$ disjoint disks by identifying them all by the identity along their boundaries. Construct a covering map $q: X \to X_n$.

*Hint: It might be helpful to think about the case $n = 2$ first.*

(c) If $n$ is even let $G = \{2k \mid k \in \mathbb{Z}/n\}$. Construct a covering map $p: Y \to X_n$ of $X_n$ such that $p_*(\pi_1(Y, \bar{x}_0)) = G$.

Solution.

(a) The space $X_n$ is a CW-complex with one 0-cell, one 1-cell, and one 2-cell. A small neighborhood $U$ of the 1-cell is homotopy equivalent to $S^1$ since both endpoints are joined to the same 0-cell. The space $X_n$ is covered by the open set $U$ and the 2-cell, which is also open. Their intersection is path-connected. Let the loop that goes around the edge once in positive direction be $\gamma$ and $a = [\gamma] \in \pi_1(U)$. Then by van Kampen’s theorem the fundamental group of $X_n$ is generated by $a$ (since the 2-cell is simply connected, there are no additional generators) has the relation $a^n = 1$. Thus $\pi_1(X_n) \cong \langle a \mid a^n \rangle \cong \mathbb{Z}/n$.

(Alternatively, cite the exercise class where we computed the effect on the fundamental group of attaching a 2-cell to a CW-complex.)

(b) Map the first disk to the 2-cell of $X_n$. Map the $k$-th disk to the 2-cell in the same way but first turn by an angle of $2\pi k/n$. This is an $n$-fold covering as each point in the interior of the 2-cell has an open neighborhood that is evenly covered by $n$ copies of this neighborhood in the $n$ disks. A point on a 1-cell or 0-cell has a neighborhood that is homeomorphic to $n$ half-disks glued together along their bounding diameter. They are evenly covered by $n$ homeomorphic sets in $X$, one for each edge of $X$.

A precise construction is, for example, the following: Let $D \subseteq \mathbb{C}$ be the closed unit disk in the complex plane. Define the map $q_k: D \to D, x \mapsto e^{2\pi i k/n}x$ for each $k = 1, \ldots, n$. Denote the $n$-gon by $P$ and fix a homeomorphism $h: D \to P$. Let $\Phi: P \to X_n$ and $\Psi: \bigsqcup_{i=1}^n D \to X$ be the canonical quotient maps. The maps $q_k$ induce a map $q': \bigsqcup_{i=1}^n D \to X_n$ by mapping $x$ in the $k$-th copy of $D$ to $\Phi(h(q_k(x)))$.

Since $q'(x) = q'(y)$ if $\Phi(x) = \Phi(y)$ for $x, y \in \bigsqcup_{i=1}^n D$, the map $q'$ induces a well-defined map on the quotient $q: X \to X_n$. 

(c) The space $Y$ is a quotient of $X$ where all even numbered disks and all odd numbered disks are identified with one another, respectively. Let $Y$ be $S^2$ with the equator partitioned into $n$ equal parts such that every other part is identified. More precisely, let $S^2$ be the unit sphere in $\mathbb{C} \times \mathbb{R}$. Then $Y = S^2/\sim$, where the equivalence relation $\sim$ is generated by $(z,0) \sim (e^{2\pi i \frac{2}{n}} z,0)$.

The map $p$ maps the upper hemisphere homeomorphically onto the 2-cell of $X_n$ and maps the lower hemisphere in the same way after first turning by an angle of $\frac{2\pi}{n}$.

A different way to think about coverings of CW-complexes is the following: the group $G$ has index 2 in $\mathbb{Z}/n$, that is, the quotient group $(\mathbb{Z}/n)/G$ has two elements. Thus, $Y$ must be a double covering of $X_n$. The space $Y$ can be constructed as a CW-complex by lifting the cell complex structure on $X_n$.

Since $X_n$ has one 0-cell, $Y$ has two 0-cells. Likewise, $Y$ has two 1-cells $e_1$ and $e_2$ that both connect the two 0-cells and two 2-cells. Both 2-cells are glued into the circle determined by $e_1$ and $e_2$ via a map that wraps $m$ times around this circle, where $n = 2m$. Notice that $\pi_1(Y) \cong \mathbb{Z}/m$. Since $p_*$ is always injective, its image is isomorphic to $\mathbb{Z}/m$. Thus, $p_*$ is always injective, its image is isomorphic to $\mathbb{Z}/m < \mathbb{Z}/n$.

The covering map sends both 0-cells to the 0-cell of $X_n$. It sends the 1-cells homomorphically to the 1-cell of $X_n$. This results in a map that wraps the circle determined by $e_1$ and $e_2 n$ times around the 1-cell of $X_n$. Since in $X_n$ there is a 2-cell glued into the 1-cell by wrapping around $n$ times, this map is nullhomotopic and can be extended to the entire disk. This defines the covering map on all of $Y$. 
