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11.12.2014

Practice Test – Topology

The midterm exam will take place on December 17, 2.15 p.m. in MA 313. Please be there a couple of minutes early. You may bring an A4 sheet of **handwritten** notes. You will have 80 minutes to complete the exam.

Exercise 1 **10 points**

Let

- $X = \mathbb{R}^3 \setminus (\{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\})$, that is, X is $\mathbb{R}^3$ with a line and a circle around this line removed,
- $Y = \mathbb{R}^2/{\sim}$, where $(x,y) \sim (x',y')$ if and only if $y = y'$ and $x - x' \in \mathbb{Z}$, and
- let $Z = S^1 \times S^1$ be the torus.

Decide which pairs of spaces X, Y, Z are homotopy equivalent, that is, whether $X \simeq Y$, $Y \simeq Z$, and $X \simeq Z$.

Solution. We first show that $Y \simeq S^1$. Denote by $p\colon \mathbb{R}^2 \to Y$ the canonical quotient map, and let $H\colon Y \times [0,1] \to Y, ([x,y],t) \mapsto p(x,(1-t)y)$. This map is well-defined and a deformation retraction from Y onto $p(\mathbb{R} \times \{0\}) \cong S^1$. Thus Y and Z are not homotopy equivalent since $\pi_1(Y) \cong \mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(Z)$.

We will now show that $X \simeq Z$. By transitivity of homotopy equivalence it also follows that $X \not\simeq Y$. By applying the inverse stereographic projection, X is homeomorphic to S^3 with the standard Hopf link $L = \{(x_1,\dots,x_4) \in S^3 \mid x_3 = x_4 = 0\} \cup \{(x_1,\dots,x_4) \in S^3 \mid x_1 = x_2 = 0\}$ removed. It was shown in the lecture that $S^3 \setminus L \cong \mathbb{R} \times S^1 \times S^1 \simeq S^1 \times S^1$.

Exercise 2 **10 points**

Let $K = ([0,1] \times [0,1])/{\sim}$, where the equivalence relation $\sim$ is generated by $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$, that is, K is the Klein bottle. Denote by $p\colon [0,1] \times [0,1] \to K$ the corresponding quotient map. Let $U = p((\frac{1}{4},\frac{3}{4}) \times [0,1])$ and $V = p([0,\frac{1}{3}) \times [0,1]) \cup p((\frac{2}{3},1] \times [0,1])$. Check that the prerequisites of van Kampen's theorem are satisfied, and use it to compute the fundamental group of K from those of U and V.

Solution. The set $(\frac{1}{4},\frac{3}{4}) \times [0,1] = ((\frac{1}{4},\frac{3}{4}) \times \mathbb{R}) \cap ([0,1] \times [0,1])$ is open in $[0,1] \times [0,1]$ by definition of subspace topology. By definition of quotient topology $U = p((\frac{1}{4},\frac{3}{4}) \times [0,1])$ is also open. The same reasoning applies to V since $[0,\frac{1}{3}) \times [0,1] = (-1,\frac{1}{3}) \times \mathbb{R}) \cap ([0,1] \times [0,1])$ and $(\frac{2}{3},1] \times [0,1] = ((\frac{2}{3},2) \times \mathbb{R}) \cap ([0,1] \times [0,1])$.

The intersection $U \cap V = p((\frac{1}{4},\frac{1}{3}) \times [0,1]) \cup p((\frac{2}{3},\frac{3}{4}) \times [0,1])$ has at most two path-components as continuous image of two path-connected sets. In fact, $U \cap V$ has only one path-component. To see this it suffices to find a path from a point in $p((\frac{1}{4},\frac{1}{3}) \times [0,1])$ to a point in $p((\frac{2}{3},\frac{3}{4}) \times [0,1])$. Let $\gamma_1\colon [0,1] \to [0.1] \times [0,1], t \mapsto (\frac{3}{10},(1-t)\frac{1}{2})$ and $\gamma_2\colon [0,1] \to [0.1] \times [0,1], t \mapsto (\frac{7}{10},t\frac{1}{2})$,. Then $(p \circ \gamma_1) \cdot (p \circ \gamma_2)$ is such a continuous path.

The set U deformation retracts to $p(\{\frac{1}{2}\} \times [0,1]) \cong S^1$, and V deformation retracts to $p(\{0\} \times [0,1]) \cong S^1$. The intersection deformation retracts to $p((\{\frac{3}{10}\} \times [0,1]) \cup (\{\frac{7}{10}\} \times [0,1])) \cong S^1$. Thus $\pi_1(U) \cong \langle a \rangle$, $\pi_1(V) \cong \langle b \rangle$, and $\pi_1(U \cap V) \cong \langle c \rangle$. Denote by $\iota_U\colon U \cap V \to U$ and $\iota_V\colon U \cap V \to V$ the inclusions. Then $(\iota_U)_*(c) = a^2$ and $(\iota_V)_*(c) = b^2$.

To see this we construct representatives of a, b, and c. Let $\gamma_a\colon [0,1] \to U, t \mapsto p(\frac{1}{2}, t)$, $\gamma_b\colon [0,1] \to V, t \mapsto p(0,t)$ and $\gamma_c\colon [0,1] \to U \cap V$ with $\gamma_c(t) = p(\frac{3}{10}, 2t)$ for $t \leq \frac{1}{2}$ and $\gamma_c(t) = p(\frac{7}{10}, 2t-1)$ for $t \geq \frac{1}{2}$. Then $a = [\gamma_a]$, $b = [\gamma_b]$, and $c = [\gamma_c]$. A homotopy from γ_c to $\gamma_a \cdot \gamma_a$ is given by $H_a\colon [0,1] \times [0,1] \to U$ with $H_a(s,t) = ((1-t)\frac{3}{10} + t\frac{1}{2}, 2s)$ for $s \leq \frac{1}{2}$ and $H_a(s,t) = ((1-t)\frac{7}{10} + t\frac{1}{2}, 2s-1)$ for $s \geq \frac{1}{2}$. A homotopy from γ_c to $\gamma_b \cdot \gamma_b$ is given by $H_b\colon [0,1] \times [0,1] \to V$ with $H_b(s,t) = ((1-t)\frac{3}{10}, 2s)$ for $s \leq \frac{1}{2}$ and $H_b(s,t) = ((1-t)\frac{7}{10}, 2s-1)$ for $s \geq \frac{1}{2}$. By van Kampen's theorem $\pi_1(K) \cong \langle a, b \mid a^2 = b^2 \rangle$.

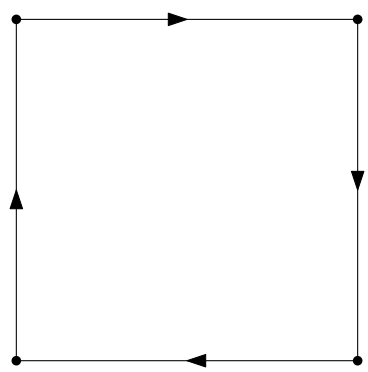
Exercise 3 **10 points**

Let X_n be the space obtained from an n-gon (that is a 2-cell bounded by n 1-cells) by identifying all edges with the same orientation around the n-gon.

(a) Argue that $\pi_1(X_n, x_0) \cong \mathbb{Z}/n$.

(b) Let X be the space that is obtained from n disjoint disks by identifying them all by the identity along their boundaries. Construct a covering map $q\colon X \to X_n$.
Hint: It might be helpful to think about the case $n = 2$ first.

(c) If n is even let $G = \{2k \mid k \in \mathbb{Z}/n\}$. Construct a covering map $p\colon Y \to X_n$ of X_n such that $p_*(\pi_1(Y, \widetilde{x}_0)) = G$.



Solution.

(a) The space X_n is a CW-complex with one 0-cell, one 1-cell, and one 2-cell. A small neighborhood U of the 1-cell is homotopy equivalent to S^1 since both endpoints are joined to the same 0-cell. The space X_n is covered by the open set U and the 2-cell, which is also open. Their intersection is path-connected. Let the loop that goes around the edge once in positive direction be γ and $a = [\gamma] \in \pi_1(U)$. Then by van Kampen's theorem the fundamental group of X_n is generated by a (since the 2-cell is simply connected, there are no additional generators) has the relation $a^n = 1$. Thus $\pi_1(X_n) \cong \langle a \mid a^n \rangle \cong \mathbb{Z}/n$. (Alternatively, cite the exercise class where we computed the effect on the fundamental group of attaching a 2-cell to a CW-complex.)

(b) Map the first disk to the 2-cell of X_n. Map the k-th disk to the 2-cell in the same way but first turn by an angle of $2\pi\frac{k-1}{n}$. This is an n-fold covering as each point in the interior of the 2-cell has an open neighborhood that is evenly covered by n copies of this neighborhood in the n disks. A point on a 1-cell or 0-cell has a neighborhood that is homeomorphic to n half-disks glued together along their bounding diameter. They are evenly covered by n homeomorphic sets in X, one for each edge of X. A precise construction is, for example, the following: Let $D \subseteq \mathbb{C}$ be the closed unit disk in the complex plane. Define the map $q_k\colon D \to D, x \mapsto e^{2\pi i \frac{k-1}{n}} x$ for each $k = 1, \ldots, n$. Denote the n-gon by P and fix a homeomorphism $h\colon D \to P$. Let $\Phi\colon P \to X_n$ and $\Psi\colon \bigsqcup_{i=1}^n D \to X$ be the canonical quotient maps. The maps q_k induce a map $q'\colon \bigsqcup_{i=1}^n D \to X_n$ by mapping x in the k-th copy of D to $\Phi(h(q_k(x)))$. Since $q'(x) = q'(y)$ if $\Phi(x) = \Phi(y)$ for $x, y \in \bigsqcup_{i=1}^n D$, the map q' induces a well-defined map on the quotient $q\colon X \to X_n$.

(c) The space Y is a quotient of X where all even numbered disks and all odd numbered disks are identified with one another, respectively. Let Y be S^2 with the equator partitioned into n equal parts such that every other part is identified. More precisely, let S^2 be the unit sphere in $\mathbb{C} \times \mathbb{R}$. Then $Y = S^2/\sim$, where the equivalence relation $\sim$ is generated by $(z, 0) \sim (e^{2\pi i \frac{2}{n}} z, 0)$.

The map p maps the upper hemisphere homeomorphically onto the 2-cell of X_n and maps the lower hemisphere in the same way after first turning by an angle of $\frac{2\pi}{n}$.

A different way to think about coverings of CW-complexes is the following: the group G has index 2 in $\mathbb{Z}/n$, that is, the quotient group $(\mathbb{Z}/n)/G$ has two elements. Thus, Y must be a double covering of X_n. The space Y can be constructed as a CW-complex by lifting the cell complex structure on X_n. Since X_n has one 0-cell, Y has two 0-cells. Likewise, Y has two 1-cells e_1 and e_2 that both connect the two 0-cells and two 2-cells. Both 2-cells are glued into the circle determined by e_1 and e_2 via a map that wraps m times around this circle, where $n = 2m$. Notice that $\pi_1(Y) \cong \mathbb{Z}/m$. Since p_* is always injective, its image is isomorphic to $\mathbb{Z}/m < \mathbb{Z}/n$.

The covering map sends both 0-cells to the 0-cell of X_n. It sends the 1-cells homomorphically to the 1-cell of X_n. This results in a map that wraps the circle determined by e_1 and e_2 n times around the 1-cell of X_n. Since in X_n there is a 2-cell glued into the 1-cell by wrapping around n times, this map is nullhomotopic and can be extended to the entire disk. This defines the covering map on all of Y.