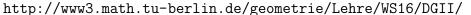
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WS 16

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 1

(Manifolds)

due 01.11.2016

Exercise 1 3 points

Let X be a topological space, $x \in X$ and $n \ge 0$. Show that the following statements are equivalent:

- i) There is a neighborhood of x which is homeomorphic to \mathbb{R}^n .
- ii) There is a neighborhood of x which is homeomorphic to an open subset of \mathbb{R}^n .

Exercise 2 6 points

Let $n \in \mathbb{N}$ and $X := \mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{R}P^n = X/_{\sim}$ with equivalence relation given by

$$x \sim y : \iff x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the *n*-dimensional real projective space. Let $\pi : X \to \mathbb{R}P^n$ denote the canonical projection $x \mapsto [x]$. For $i = 0, \ldots, n$, we define $U_i := \pi(\{x \in X \mid x_i \neq 0\})$ and $\varphi_i : U_i \to \mathbb{R}^n$ by

$$[x_0,\ldots,x_n]\mapsto (x_0/x_i,\ldots,\widehat{x_i},\ldots,x_n/x_i).$$

Show that

- a) π is an open map, i.e. maps open sets in X to open sets in $\mathbb{R}P^n$,
- b) the maps φ_i are well-defined and $\{(U_i, \varphi_i)\}_{i \in I}$ is a smooth atlas of $\mathbb{R}P^n$,
- c) $\mathbb{R}P^n$ is compact. Hint: Note that the restriction of π to \mathbb{S}^n is surjective.

Exercise 3 6 points

Let M and N be topological manifolds of dimension m and n, respectively. Show that their Cartesian product M × N is a topological manifold of dimension m + n. Show further that, if $\{(U_{\alpha}, \varphi_{\alpha})\}_{\alpha \in A}$ is a smooth atlas of M and $\{(V_{\beta}, \psi_{\beta})\}_{\beta \in B}$ is a smooth atlas of N, then $\{(U_{\alpha} \times V_{\beta}, \varphi_{\alpha} \times \psi_{\beta})\}_{(\alpha,\beta) \in A \times B}$ is a smooth atlas of M × N. Here $\varphi_{\alpha} \times \psi_{\beta} \colon U_{\alpha} \times V_{\beta} \to \varphi_{\alpha}(U_{\alpha}) \times \psi_{\beta}(V_{\beta})$ is defined by $\varphi_{\alpha} \times \psi_{\beta}(p,q) := (\varphi_{\alpha}(p), \psi_{\beta}(q))$.