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http://www3.math.tu-berlin.de/geometrie/Lehre/WS16/DGII/ WS 16

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 1

(Manifolds)

due 01.11.2016

Exercise 1 — 3 points

Let X be a topological space, $x \in \mathrm{X}$ and $n \geq 0$. Show that the following statements are equivalent:

i) There is a neighborhood of x which is homeomorphic to $\mathbb{R}^n$.
ii) There is a neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 2 — 6 points

Let $n \in \mathbb{N}$ and $\mathrm{X} := \mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{RP}^n = \mathrm{X}/_\sim$ with equivalence relation given by

$$x \sim y :\Longleftrightarrow x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the n-dimensional *real projective space*. Let $\pi\colon \mathrm{X} \to \mathbb{RP}^n$ denote the *canonical projection* $x \mapsto [x]$. For $i = 0, \ldots, n$, we define $U_i := \pi(\{x \in X \mid x_i \neq 0\})$ and $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$[x_0, \ldots, x_n] \mapsto (x_0/x_i, \ldots, \widehat{x_i}, \ldots, x_n/x_i).$$

Show that

a) π is an *open map*, i.e. maps open sets in X to open sets in $\mathbb{RP}^n$,
b) the maps φ_i are well-defined and $\{(U_i, \varphi_i)\}_{i\in I}$ is a smooth atlas of $\mathbb{RP}^n$,
c) $\mathbb{RP}^n$ is compact. **Hint:** Note that the restriction of π to $\mathbb{S}^n$ is surjective.

Exercise 3 — 6 points

Let M and N be topological manifolds of dimension m and n, respectively. Show that their Cartesian product $\mathrm{M} \times \mathrm{N}$ is a topological manifold of dimension $m + n$. Show further that, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is a smooth atlas of M and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ is a smooth atlas of N, then $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha,\beta)\in A\times B}$ is a smooth atlas of $\mathrm{M} \times \mathrm{N}$. Here $\varphi_\alpha \times \psi_\beta\colon U_\alpha \times V_\beta \to \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta)$ is defined by $\varphi_\alpha \times \psi_\beta(p, q) := (\varphi_\alpha(p), \psi_\beta(q))$.