Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 2

(Manifolds, diffeomorphisms)

due 08.11.2016

Exercise 1 **7 points**

Let $\mathbb{R}^n/\mathbb{Z}^n$ denote the quotient space $\mathbb{R}^n/_\sim$ with equivalence relation given by

$$x \sim y :\Leftrightarrow x - y \in \mathbb{Z}^n.$$

Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \mapsto [x]$ denote the canonical projection. Show:

a) π is a *covering map*, i.e. a continuous surjective map such that each point $p \in \mathbb{R}^n/\mathbb{Z}^n$ has a open neighborhood V such that $\pi^{-1}(V)$ is a disjoint union of open sets each of which is mapped by π homeomorphically to V.
b) π is an open map.
c) $\mathbb{R}^n/\mathbb{Z}^n$ is a manifold of dimension n.
d) $\{(\pi|_U)^{-1} \mid U \subset \mathbb{R}^n \textit{ open}, \;\; \pi|_U : U \to \pi(U) \textit{ bijective}\}$ is a smooth atlas of $\mathbb{R}^n/\mathbb{Z}^n$.

Exercise 2 **4 points**

Show that the following manifolds are diffeomorphic.

a) $\mathbb{R}^2/\mathbb{Z}^2$.
b) the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$.
c) the torus of revolution as a submanifold of $\mathbb{R}^3$:

$$T = \big\{((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi) \mid \varphi, \theta \in \mathbb{R}\big\}.$$

Exercise 3 **4 points**

Show that the Möbius band (without boundary)

$$\mathrm{M} = \big\{((2 + r\cos\tfrac{\varphi}{2})\cos\varphi, (2 + r\cos\tfrac{\varphi}{2})\sin\varphi, r\sin\tfrac{\varphi}{2}) \mid r \in (-\tfrac{1}{2}, \tfrac{1}{2}),\ \varphi \in \mathbb{R}\big\}$$

is a submanifold of $\mathbb{R}^3$. Show further that for each point $p \in \mathbb{R}\mathrm{P}^2$ the open set $\mathbb{R}\mathrm{P}^2 \setminus \{p\} \subset \mathbb{R}\mathrm{P}^2$ is diffeomorphic to M.