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http://www3.math.tu-berlin.de/geometrie/Lehre/WS16/DGII/ WS 16

Berlin Mathematical School

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 9

(Riemannian manifolds, isometries)

due 17.01.2017

Definition: Let $f\colon \mathrm{M} \to \mathrm{N}$ be a diffeomorphism and $X \in \Gamma(\mathrm{TM})$. Then the *push forward* $f_*X \in \Gamma(\mathrm{TN})$ *of* X is defined by $f_*X := df \circ X \circ f^{-1}$.

Exercise 1 **5 points**

Let $f\colon \mathrm{M} \to \mathrm{N}$ be a diffeomorhism and $X, Y \in \Gamma(\mathrm{TM})$. Show: $f_*[X,Y] = [f_*X, f_*Y]$.

Exercise 2 **5 points**

Let (M, g) and (N, h) be Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, respectively. Let $f\colon \mathrm{M} \to \mathrm{N}$ be an isometry. Show that $\tilde{\nabla}_{f_*X} f_*Y = f_*\left(\nabla_X Y\right)$.

Exercise 3 **5 points**

a) Show that $\langle X, Y\rangle := \frac{1}{2}\mathrm{trace}(\bar{X}^t Y)$ defines a Riemannian metric on $\mathrm{SU}(2)$.
b) Show that the left and the right multiplication by a constant g are isometries.
c) Show that $\mathrm{SU}(2)$ and the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ (with induced metric) are isometric.
Hint: $\mathrm{SU}(2) = \{\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$.