ANALYSIS AND GEOMETRY ON MANIFOLDS

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Contents

1. n-Dimensional Manifolds

1.1. **Introduction.** Informally, an n-dimensional manifold is a "space" which locally (when looked at through a microscope) looks like "flat space" $\mathbb{R}^n$.

Many important examples of manifolds M arise as certain subsets $\mathrm{M} \subset \mathbb{R}^k$, e.g.:

(1) n-dimensional affine subspaces $\mathrm{M} \subset \mathbb{R}^k$,
(2) $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots x_n^2 = 1\}$,

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(3) compact 2-dimensional submanifolds of $\mathbb{R}^3$
(4) $\mathrm{SO}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^t A = Id\}$ is a 3-dimensional submanifold of $\mathbb{R}^9$.

Flat spaces (vector spaces $\cong \mathbb{R}^n$) are everywhere. Curved manifolds come up in Stochastics, Algebraic Geometry, ..., Economics and Physics – e.g. as the configuration space of a pendulum ($\mathbb{S}^2$), a double pendulum ($\mathbb{S}^2 \times \mathbb{S}^2$) or rigid body motion ($\mathrm{SO}(3)$), or as space time in general relativity (the curved version of flat special relativity).

1.2. **Crash Course in Topology.**

Definition 1 (Topological space)**.** *A topological space is a set* M *together with a subset* $\mathcal{O} \subset \mathcal{P}(\mathrm{M})$ *(the collection of all "open sets") such that:*

(1) $\emptyset, \mathrm{M} \in \mathcal{O}$,
(2) $U_\alpha \in \mathcal{O}$, $\alpha \in I \Rightarrow \cup_{\alpha\in I} U_\alpha \in \mathcal{O}$,
(3) $U_1, \ldots, U_n \in \mathcal{O}, \Rightarrow U_1 \cap \cdots \cap U_n \in \mathcal{O}$.

Remark 1: *Usually we suppress the the collection* $\mathcal{O}$ *of open sets and just say* M *is a topological space. If several topologies and spaces are involved we use an index to make clear which topology corresponds to which space.*

Some ways to make new topological spaces out of given ones:

a) Let X be a topological space, $\mathrm{M} \subset \mathrm{X}$, then $\mathcal{O}_\mathrm{M} := \{U \cap \mathrm{M} \mid U \in \mathcal{O}_\mathrm{X}\}$ defines a topology on M – called "induced topology" or "subspace topology".
b) Let X be a topological space, M be a set, and $\pi\colon X \to \mathrm{M}$ a surjective map. Then there is a bijection between M and the set of equivalence classes of the equivalence relation on X defined by
$$x \sim y \Leftrightarrow \pi(x) = \pi(y).$$
In other words: M can be identified with the set of equivalence classes. Conversely, given an equivalence relation $\sim$ on a topological space X we can form the set of equivalence classes $\mathrm{M} = \mathrm{X}/_\sim$. The *canonical projection* $\pi\colon \mathrm{X} \to \mathrm{M}$ is the surjective map which sends $x \in \mathrm{X}$ to the corresponding equivalence class $[x]$. The *quotient topology*
$$\mathcal{O}_\mathrm{M} = \{U \subset \mathrm{M} \mid \pi^{-1}(U) \in \mathcal{O}_\mathrm{X}\}$$
turns M into a topological space. By construction π is continuous.

Exercise 1 (Product topology)**.** *Let* M *and* N *be topological spaces and define* $\mathcal{B} := \{U \times V \mid U \in \mathcal{O}_\mathrm{M}, V \in \mathcal{O}_\mathrm{N}\}$. *Show that* $\mathcal{O} := \{\cup_{U\in\mathcal{A}} U \mid \mathcal{A} \subset \mathcal{B}\}$ *is a topology on* $\mathrm{M} \times \mathrm{N}$.

Definition 2 (Continuity)**.** *Let* M, N *be topological spaces. Then* $f\colon \mathrm{M} \to \mathrm{N}$ *is called continuous if*
$$f^{-1}(U) \in \mathcal{O}_\mathrm{M} \textit{ for all } U \in \mathcal{O}_\mathrm{N}.$$

Definition 3 (Homeomorphism)**.** *A bijective map* $f\colon \mathrm{M} \to \mathrm{N}$ *between topological spaces is called a homeomorphism if* f *and* f^{-1} *are both continuous.*

Remark 2: *If* $f\colon \mathrm{M} \to \mathrm{N}$ *is a homeomorphism, then for* $U \in \mathcal{O}_\mathrm{M} \Leftrightarrow f(U) \in \mathcal{O}_\mathrm{N}$. *So two topological spaces are topologically indistinguishable, if they are homeomorphic, i.e. if there exists a homeomorphism* $f\colon \mathrm{M} \to \mathrm{N}$.

Definition 4 (Hausdorff)**.** *A topological space* M *is called Hausdorff if for all* $x, y \in \mathrm{M}$ *with* $x \neq y$ *there are open sets* $U_x, U_y \in \mathcal{O}$ *with* $U_x \cap U_y = \emptyset$.

Example: *The quotient space* $M = \mathbb{R}/\sim$ *with* $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ *is not Hausdorff.*

Definition 5 (Second axiom of countability)**.** *A topological space* M *is said to satisfy the second axiom of countability (or is called second countable), if there is a countable base of topology, i.e. there is a sequence of open sets* $U_1, U_2, U_3, \ldots \in \mathcal{O}$ *such that for every* $U \in \mathcal{O}$ *there is a subset* $I \subset \mathbb{N}$ *such that* $U = \cup_{\alpha \in I} U_\alpha$*.*

Example: *The balls of rational radius with rational center in* $\mathbb{R}^n$ *form a countable base of topology, i.e.* $\mathbb{R}^n$ *is 2nd countable.*

Remark 3: *Subspaces of second countable spaces are second countable. Hence all subsets of* $\mathbb{R}^n$ *are second countable. A similar statement holds for the Hausdorff property.*

Example: $\mathrm{M} = \mathbb{R}^2$ *with* $\mathcal{O} = \{U \times \{y\} \mid y \in \mathbb{R},\, U \in \mathcal{O}_{\mathbb{R}}\}$ *is not second countable.*

Definition 6 (Topological manifold)**.** *A topological space* M *is called an* n*-dimensional topological manifold if it is Hausdorff, second countable and for every* $p \in \mathrm{M}$ *there is an open set* $U \in \mathcal{O}$ *with* $p \in U$ *and a homeomorphism* $\varphi\colon U \to V$*, where* $V \in \mathcal{O}_{\mathbb{R}^n}$*.*

Remark 4: *A homeomorphism* $\varphi\colon U \to V$ *as above is called a (coordinate) chart of* M*.*

Exercise 2. *Let* X *be a topological space,* $x \in \mathrm{X}$ *and* $n \geq 0$*. Show that the following statements are equivalent:*

i) There is a neighborhood of x *which is homeomorphic to* $\mathbb{R}^n$*.*
ii) There is a neighborhood of x *which is homeomorphic to an open subset of* $\mathbb{R}^n$*.*

Exercise 3. *Show that a manifold* M *is locally compact, i.e. each point of* M *has a compact neighborhood.*

Exercise 4 (Connectedness)**.** *A topological space* M *is connected if the only subsets of* X *which are simultaneously open and closed are* X *and* $\emptyset$*. Moreover,* X *is called path-connected if any two points* $x, y \in \mathrm{X}$ *can be joined by a path, i.e. there is a continuous map* $\gamma\colon [a,b] \to \mathrm{X}$ *such that* $\gamma(0) = x$ *and* $\gamma(1) = y$*. Show that a manifold is connected if and only it is path-connected.*

Given two charts $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon V \to \mathbb{R}^n$, then the map $f\colon \varphi(U \cap V) \to \psi(U \cap V)$ given by $f = \psi \circ (\varphi|_{U \cap V})^{-1}$ is a homeomorphism, called the *coordinate change* or *transition map.*

Definition 7 (Atlas)**.** *An atlas of a manifold* M *is a collection of charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ *such that* $\mathrm{M} = \cup_{\alpha \in I} U_\alpha$*.*

Definition 8 (Compatible charts)**.** *Two charts* $\varphi\colon U \to \mathbb{R}^n$*,* $\psi\colon V \to \mathbb{R}^n$ *on a topological manifold* M *are called compatible if* $f\colon \varphi(U \cap V) \to \psi(U \cap V)$ *is a diffeomorphism, i.e.* f *and* f^{-1} *both are smooth.*

Example: *Consider* $\mathrm{M} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$*. Let* $B = \{y \in \mathbb{R}^n \mid \|y\| \leq 1\}$*. Define charts as follows: For* $i = 0, \ldots, n$*,*

$$U_i^{\pm} = \{x \in \mathbb{S}^2 \mid \pm x_i > 0\}, \quad \varphi_i^{\pm}, \colon U_i^{\pm} \to B, \quad \varphi_i^{\pm}(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n),$$

where the hat means omission. Check that φ_i *are homeomorphisms. So: (Since* $\mathbb{S}^n$ *as a subset of* $\mathbb{R}^{n+1}$ *is Hausdorff and second countable)* $\mathbb{S}^n$ *is an* n*-dimensional topological manifold. All* $\varphi_i^{\pm}$ *are compatible, so this atlas turns* $\mathbb{S}^n$ *into a smooth manifold.*

An atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of mutually compatible charts on M is called *maximal* if every chart (U, φ) on M which is compatible with all charts in $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is already contained in the atlas.

Definition 9 (Smooth manifold)**.** *A differentiable structure on a topological manifold* M *is a maximal atlas of compatible charts. A smooth manifold is a topological manifold together with a maximal atlas.*

Exercise 5 (Real projective space)**.** *Let $n \in \mathbb{N}$ and $\mathrm{X} := \mathbb{R}^{n+1} \setminus \{0\}$. The quotient space $\mathbb{RP}^n = \mathrm{X}/_\sim$ with equivalence relation given by*

$$x \sim y :\Longleftrightarrow x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the n-dimensional real projective space. Let $\pi \colon \mathrm{X} \to \mathbb{RP}^n$ denote the canonical projection $x \mapsto [x]$.

For $i = 0, \ldots, n$, we define $U_i := \pi(\{x \in X \mid x_i \neq 0\})$ and $\varphi_i \colon U_i \to \mathbb{R}^n$ by

$$[x_0, \ldots, x_n] \mapsto (x_0/x_i, \ldots, \widehat{x_i}, \ldots, x_n/x_i).$$

Show that

a) π is an open map, i.e. maps open sets in X to open sets in $\mathbb{RP}^n$,
b) the maps φ_i are well-defined and $\{(U_i, \varphi_i)\}_{i \in I}$ is a smooth atlas of $\mathbb{RP}^n$,
c) $\mathbb{RP}^n$ is compact. ***Hint:*** *Note that the restriction of π to $\mathbb{S}^n$ is surjective.*

Exercise 6 (Product manifolds)**.** *Let M and N be topological manifolds of dimension m and n, respectively. Show that their Cartesian product $\mathrm{M} \times \mathrm{N}$ is a topological manifold of dimension $m + n$. Show further that, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is a smooth atlas of M and $\{(V_\beta, \psi_\beta)\}_{\beta \in B}$ is a smooth atlas of N, then $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha,\beta) \in A \times B}$ is a smooth atlas of $\mathrm{M} \times \mathrm{N}$. Here $\varphi_\alpha \times \psi_\beta \colon U_\alpha \times V_\beta \to \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta)$ is defined by $\varphi_\alpha \times \psi_\beta(p, q) := (\varphi_\alpha(p), \psi_\beta(q))$.*

Exercise 7 (Torus)**.** *Let $\mathbb{R}^n/\mathbb{Z}^n$ denote the quotient space $\mathbb{R}^n/_\sim$ where the equivalence relation is given by*

$$x \sim y :\Leftrightarrow x - y \in \mathbb{Z}^n.$$

Let $\pi \colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \mapsto [x]$ denote the canonical projection. Show:

a) π is a covering map, i.e. a continuous surjective map such that each point $p \in \mathbb{R}^n/\mathbb{Z}^n$ has a open neighborhood V such that $\pi^{-1}(V)$ is a disjoint union of open sets each of which is mapped by π homeomorphically to V.
b) π is an open map.
c) $\mathbb{R}^n/\mathbb{Z}^n$ is a manifold of dimension n.
d) $\{(\pi|_U)^{-1} \mid U \subset \mathbb{R}^n \text{ open}, \; \pi|_U : U \to \pi(U) \text{ bijective}\}$ is a smooth atlas of $\mathbb{R}^n/\mathbb{Z}^n$.

Definition 10 (Smooth map)**.** *Let M and $\tilde{\mathrm{M}}$ be smooth manifolds. Then a map $f \colon \mathrm{M} \to \tilde{\mathrm{M}}$ is called smooth if for every chart (U, φ) of M and every chart (V, ψ) of $\tilde{\mathrm{M}}$ the map*

$$\varphi(f^{-1}(V) \cap U) \to \psi(V), \quad x \mapsto \psi(f(\varphi^{-1}(x)))$$

is smooth.

Definition 11 (Diffeomorphism)**.** *Let M and $\tilde{\mathrm{M}}$ be smooth manifolds. Then a bijective map $f \colon \mathrm{M} \to \tilde{\mathrm{M}}$ is called a diffeomorphism if both f and f^{-1} are smooth.*

One important task in Differential Topology is to classify all smooth manifolds up to diffeomorphism.

Example: *Every connected one-dimensional smooth manifold is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Examples of 2-dimensional manifolds: [pictures missing: compact genus 0,1,2,... Klein bottle, or torus with holes (non-compact)] - gets much more complicated already. For 3-dimensional manifolds there is no list.*

Exercise 8. *Show that the following manifolds are diffeomorphic.*

a) $\mathbb{R}^2/\mathbb{Z}^2$.
b) the product manifold $\mathbb{S}^1 \times \mathbb{S}^1$.

c) the torus of revolution as a submanifold of $\mathbb{R}^3$*:*

$$T = \Big\{((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi) \mid \varphi, \theta \in \mathbb{R}\Big\}.$$

1.3. **Submanifolds.**

Definition 12 (Submanifold)**.** *A subset* $\mathrm{M} \subset \tilde{\mathrm{M}}$ *in a* k*-dimensional smooth manifold* $\tilde{\mathrm{M}}$ *is called an* n*-dimensional submanifold if for every point* $p \in \mathrm{M}$ *there is a chart* $\varphi\colon U \to V$ *of* $\tilde{\mathrm{M}}$ *with* $p \in U$ *such that*

$$\varphi(U \cap \mathrm{M}) = V \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^k.$$

Let us briefly restrict attention to $\tilde{\mathrm{M}} = \mathbb{R}^k$.

Theorem 1. *Let* $\mathrm{M} \subset \mathbb{R}^k$ *be a subset. Then the following are equivalent:*

a) M *is an* n*-dimensional submanifold,*
b) locally M *looks like the graph of a map from* $\mathbb{R}^n$ *to* $\mathbb{R}^{k-n}$*, which means: For every point* $p \in \mathrm{M}$ *there are open sets* $V \subset \mathbb{R}^n$ *and* $W \subset \mathrm{M}$*,* $W \ni p$*, a smooth map* $f\colon V \to \mathbb{R}^{k-n}$ *and a coordinate permutation* $\pi\colon \mathbb{R}^k \to \mathbb{R}^k$*,* $\pi(x_1, ..., x_k) = (x_{\sigma_1}, ..., x_{\sigma_k})$ *such that*

$$\pi(W) = \{(x, f(x)) \mid x \in V\},$$

c) locally M *is the zero set of some smooth map into* $\mathbb{R}^{k-n}$*, which means: For every* $p \in \mathrm{M}$ *there is an open set* $U \subset \mathbb{R}^k$*,* $U \ni p$ *and a smooth map* $g\colon U \to \mathbb{R}^{k-n}$ *such that*

$$\mathrm{M} \cap U = \{x \in U \mid g(x) = 0\}$$

and the Jacobian $g'(x)$ *has full rank for all* $x \in \mathrm{M}$*,*
d) locally M *can be parametrized by open sets in* $\mathbb{R}^n$*, which means: For every* $p \in \mathrm{M}$ *there are open sets* $W \subset \mathrm{M}$*,* $W \ni p$*,* $V \subset \mathbb{R}^n$ *and a smooth map* $\psi\colon V \to \mathbb{R}^k$ *such that* ψ *maps* V *bijectively onto* W *and* $\psi'(x)$ *has full rank for all* $x \in V$*.*

Proof. First, recall two theorems from analysis:

- The *inverse function theorem*: Let $U \subset \mathbb{R}^n$ be open, $p \in U$, $f\colon U \to \mathbb{R}^n$ continuously differentiable, $f'(p) \neq 0$. Then there is a an open' subset $\tilde{U} \subset U$, $\tilde{U} \ni p$ and an open subset $V \subset \mathbb{R}^n$, $V \ni f(p)$ such that
  (1) $f|_{\tilde{U}} : \tilde{U} \to V$ is bijective,
  (2) $f^{-1}\colon V \to \tilde{U}$ is continuously differentiable.
  We have $(f^{-1})'(q) = f'(f^{-1}(q))^{-1}$ for all $q \in V$. We in fact need a version where 'continuously differentiable' is replaced by $\mathscr{C}^\infty$. Let us prove the $\mathscr{C}^2$ version. Then all the partial derivatives of first order for f^{-1} are entries of $(f^{-1})'$. So we have to prove that $q \mapsto (f^{-1})'(q) = (f')^{-1}(f^{-1}(q))$ is continuously differentiable. This follows from the smoothness of the map $\mathrm{GL}(n, \mathbb{R}) \ni A \mapsto A^{-1} \in \mathrm{GL}(n, \mathbb{R})$ (Cramer's rule), the chain rule and the fact that $f'\colon \tilde{U} \to \mathbb{R}^{n\times n}$ is continuously differentiable. The general case can be done by induction.
- The *implicit function theorem* ($\mathscr{C}^\infty - version$): Let $U \subset \mathbb{R}^k$ be open, $p \in U$, $g\colon U \to \mathbb{R}^{k-n}$ smooth, $g(p) = 0$, $g'(p)$ is surjective. Then, after reordering the coordinates of $\mathbb{R}^k$, we find open subsets $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^{n-k}$ such that $(p_1, \ldots, p_n) \in V$ and $(p_{n+1}, \ldots, p_k) \in W$ and $V \times W \subset U$. Moreover, there is a smooth map $f\colon V \to W$ such that $\{q \in V \times W \mid g(q) = 0\} = \{(x, f(x)) \mid x \in V\}$.

Proof of Theorem 1. $b) \Rightarrow a)$: Let $p \in \mathrm{M}$. By $b)$ after reordering coordinates in $\mathbb{R}^k$ we find open sets $V \in \mathbb{R}^n$, $W \subset \mathbb{R}^{k-n}$ such that $p \in V \times W$ and we find a smooth map $f\colon V \to W$

such that $V \times W \cap \mathrm{M} = \{(x, f(x)) \mid x \in V\}$. Then $\varphi\colon V \times W \to \mathbb{R}^k, (x,y) \mapsto (x, y - f(x))$ is a diffeomorphism and $\varphi(\mathrm{M} \cap (V \times W)) \subset \mathbb{R}^n \times \{0\}$. [picture missing]
a) $\Rightarrow$ *c)*: Let $p \in \mathrm{M}$. By *a)* we find an open $U \in \mathbb{R}^k$, $U \ni p$ and a diffeomorphism $\varphi\colon U \to \hat{U} \subset \mathbb{R}^k$ such that $\varphi(U \cap \mathrm{M}) \subset \mathbb{R}^n \times \{0\}$. Now define $g\colon U \to \mathbb{R}^{k-n}$ to be the last $k-n$ component functions of φ, i.e. $\varphi = (\varphi_1, \ldots, \varphi_n, g_1, \ldots, g_{k-n})$. Then $\mathrm{M} \cap (V \times W) = g^{-1}(\{0\})$. For $q \in V \times W$ we have

$$\varphi'(q) = \begin{pmatrix} * \\ \vdots \\ * \\ g_1'(q) \\ \vdots \\ g_{k-n}'(q) \end{pmatrix}.$$

Hence g' has rank $k-n$. *c)* $\Rightarrow$ *b)* is just the implicit function theorem. Let us look at *b)* $\Rightarrow$ *d)*. Let $p \in \mathrm{M}$. After reordering the coordinates by *b)* we have an open neighborhood of p of the form $V \times W$ and a smooth map $f\colon V \to W$ such that $\mathrm{M} \cap (V \times W) = \{(x, f(x)) \mid x \in V\}$. Now define $\psi\colon V \to \mathbb{R}^k$ by $\psi(x) = (x, f(x))$. Then ψ is smooth

$$\psi'(x) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^n} \\ f'(x) \end{pmatrix}$$

So $\psi'(x)$ has rank n for all $x \in V$. Moreover, $\psi(V) = \mathrm{M} \times (V \times W)$. *d)* $\Rightarrow$ *b)*: Let $p \in \mathrm{M}$. Then by *d)* there are open sets $\hat{V} \subset \mathbb{R}^n$, $U \subset \mathbb{R}^k$, $U \ni p$ and a smooth map $\psi\colon \hat{V} \to \mathbb{R}^k$ such that $\psi(\hat{V}) = \mathrm{M} \cap U$ such that rank $\psi'(x)$ is n for all $x \in \hat{V}$. After reordering the coordinates on $\mathbb{R}^k$ we can assume that $\psi = (\phi, \hat{f})^t$ with $\phi\colon \hat{V} \to \mathbb{R}^n$ with $\det \phi'(x_0) \neq 0$, where $\psi(x_0) = p$. Passing to a smaller neighborhood $V \subset \hat{V}$, $V \ni p$, we then achieve that $\phi\colon V \to \phi(V)$ is a diffeomorphism (by the inverse function theorem). Now for all $y \in \phi(V)$ we have

$$\psi(\phi^{-1}(y)) = \begin{pmatrix} \phi(\phi^{-1}(y)) \\ \hat{f}(\phi^{-1}(y)) \end{pmatrix} =: \begin{pmatrix} y \\ f(\phi^{-1}(y)) \end{pmatrix}$$

□

1.4. Examples of submanifolds in $\mathbb{R}^k$.

a) $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ is an n-dimensional submanifold (a *hypersurface*) of $\mathbb{R}^{n+1}$, because $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid g(x) = 0\}$, where $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $g(x) = x_1^2 + x_2^2 + \cdots x_{n+1}^2 - 1$. We have to check that $g'(x)$ has rank 1 on $g^{-1}(\{0\})$: We have $g'(x) = 2x \neq 0$ for $x \neq 0$.
b) $\mathrm{O}(n) \subset \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$, $\mathrm{O}(n) = \{A \in \mathbb{R}^{n \times n} \mid A^t A = I\}$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n(n-1)/2$. Define $g\colon \mathbb{R}^{n \times n} \to \mathrm{Sym}(n) = \mathbb{R}^{n(n-1)/2}$ by $g(A) = A^t A - I$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ * & a_{22} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{pmatrix}$$

Entries above and including the diagonal: $n + (n+1) + \cdots + 2 + 1 = n(n-1)/2$. Need to check that $g'(A)\colon \mathbb{R}^{n \times n} \to \mathrm{Sym}(n)$ is surjective for all $A \in \mathrm{O}(n)$.
Interlude: Derivatives of maps $f\colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^k$ open. $f'(p)\colon \mathbb{R}^k \to \mathbb{R}^m$ linear.

How to calculate $f'(p)X$ for $X \in \mathbb{R}^k$? Choose smooth $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$ such that $\gamma(0) = p$ and $\gamma'(0) = X$. Then by the chain rule

$$(f \circ \gamma)'(0) = f'(\gamma(0))\gamma'(0) = f'(p)X.$$

So let $A \in \mathrm{O}(n)$, $X \in \mathbb{R}^{n\times n}$, $B\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ with $B(0) = A$, $B'(0) = X$ (e.g. $B(t) = A + tX$). Then

$$\begin{aligned} g'(A)X &= \tfrac{d}{dt}\Big|_{t=0} g(B(t)) = \tfrac{d}{dt}\Big|_{t=0} [B(t)^t B(t) - I] \\ &= (B^t)'(0)B(0) + B^t(0)B'(0) = X^t A + A^t X. \end{aligned}$$

To check that $g'(A)$ is surjective, let $Y \in \mathrm{Sym}(n)$ be arbitrary. So $Y \in \mathbb{R}^{n\times n}$, $Y^t = Y$. There is $X \in \mathbb{R}^{n\times n}$ with $X^t A + A^t X = Y$, e.g. $X = \frac{1}{2}AY$: $\rightsquigarrow$

$$X^t A + A^t X = \tfrac{1}{2}(Y^t A^t A + A^t A Y) = Y.$$

So $\mathrm{O}(n)$ is a submanifold dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

c) Consider the set $G_k(\mathbb{R}^n)$ the set of k-dimensional linear subspaces of $\mathbb{R}^n$. We represent a linear subspace $U \subset \mathbb{R}^k$ by the orthogonal projection $P_U \in \mathbb{R}^{n\times n}$ onto U. The map P_U is defined by

$$P_U|_U = \mathrm{Id}_U, \quad P_U|_{U^\perp} = 0. \tag{1.1}$$

P_U has the following properties:

$$P_U^2 = P_U, \quad P_U^* = P_U, \quad \mathrm{tr}\, P_U = \dim U.$$

In the decomposition $\mathbb{R}^n = U \oplus U^\perp$, we have

$$P_U = \begin{pmatrix} Id_U & 0 \\ 0 & 0 \end{pmatrix}.$$

Conversely: If $P^* = P$, then there is an orthonormal basis of $\mathbb{R}^n$ with respect to which P is diagonal.

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

If further $P^2 = P$, then $\lambda_i^2 = \lambda_i \Leftrightarrow \lambda_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$. After reordering the basis we have

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

for some $k < n$. So P is the orthogonal projection onto a k-dimensional subspace with $k = \mathrm{tr}\, P$. Thus we have

$$\mathrm{G}_k(\mathbb{R}^n) = \{P \in \mathrm{End}(\mathbb{R}^n) \mid P^2 = P,\, P^* = P,\, \mathrm{trace}\, P = k\}.$$

We fix a k-dimensional subspace V and define

$$W_V := \{L \in \mathrm{End}(\mathbb{R}^n) \mid P_V \circ L|_V \textit{ invertible}\}.$$

Since W_V is open, the intersection $\mathrm{G}_k(\mathbb{R}^n) \cap W_V$ is open in the subspace topology.

Fix a k-dimensional subspace $V \subset \mathbb{R}^n$. Then a k-dimensional subspace $U \subset \mathbb{R}^n$ 'close' to V is the graph of a linear map $Y \in \mathrm{Hom}(V, V^\perp)$: With respect to the splitting $\mathbb{R}^n = V \oplus V^\perp$,

$$U = \mathrm{Im} \begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = \{(x, Yx) \mid x \in V\}.$$

The orthogonal complement $U^\perp$ of U is then parametrized over $V^\perp$ by $(-Y^*, \mathrm{Id}_{V^\perp})$: For $x \in V$ and $y \in V^\perp$ we have

$$\left\langle \begin{pmatrix} x \\ Yx \end{pmatrix}, \begin{pmatrix} -Y^*y \\ y \end{pmatrix} \right\rangle = \langle x, -Y^*y\rangle + \langle x, Yy\rangle = 0.$$

Since rank $(-Y, \mathrm{Id}_{V^\perp})$ is $n-k$, we get

$$U^\perp = \mathrm{Im} \begin{pmatrix} -Y^* \\ \mathrm{Id}_{V^\perp} \end{pmatrix}.$$

Further, since the corresponding orthogonal projection P_U is symmetric we can write

$$P_U = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix},$$

with $A^* = A$, $B^* = B$. Explicitly $A = P_V \circ S|_V$, $B = P_{V^\perp} \circ S|_V$ and $C = P_{V^\perp} \circ S|_{V^\perp}$.

From Equation (1.1) we get

$$\begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = P_U \begin{pmatrix} \mathrm{Id}_V \\ Y \end{pmatrix} = \begin{pmatrix} A + B^*Y \\ B + CY \end{pmatrix}, \quad 0 = P_U \begin{pmatrix} -Y^* \\ \mathrm{Id}_{V^\perp} \end{pmatrix} = \begin{pmatrix} -AY^* + B^* \\ -BY^* + C \end{pmatrix}.$$

In particular, $Y^* = A^{-1}B^*$ and, since A is self-adjoint,

$$Y = BA^{-1}. \tag{1.2}$$

If we plug this relation into the equation $\mathrm{Id}_V = A + B^*Y$ we get $\mathrm{Id}_V = A(\mathrm{Id}_V + Y^*Y)$. Since $\langle Y^*Yx, x\rangle = \langle Yx, Yx\rangle \geq 0$ the map $\mathrm{Id}_V + Y^*Y$ is always invertible. This yields $A = (\mathrm{Id}_V + Y^*Y)^{-1}$. In particular, $P_U \in W_V \cap \mathrm{G}_k(\mathbb{R}^n)$. Further, since $AY^* = B^*$, we get that $B = Y(\mathrm{Id}_V + Y^*Y)^{-1}$ and, together with $C = BY^*$, $C = Y(\mathrm{Id}_V + Y^*Y)^{-1}Y^*$. Hence

$$P_U = \begin{pmatrix} (\mathrm{Id}_V + Y^*Y)^{-1} & (\mathrm{Id}_V + Y^*Y)^{-1}Y^* \\ Y(\mathrm{Id}_V + Y^*Y)^{-1} & Y(\mathrm{Id}_V + Y^*Y)^{-1}Y^* \end{pmatrix} \in W_V \cap \mathrm{G}_k(\mathbb{R}^n). \tag{1.3}$$

Equation (1.3) actually defines a smooth map $\phi\colon \mathrm{Hom}(V, V^\perp) \to W_V \cap \mathrm{G}_k(\mathbb{R}^n)$ with left inverse given by Equation (1.2), which is smooth on W_V. Hence ϕ is surjective and has full rank. Thus $\mathrm{G}_k(\mathbb{R}^n)$ is locally parametrized by $\mathrm{Hom}(V, V^\perp) \cong \mathbb{R}^{k\cdot(n-k)}$.

Theorem 2. *The Grassmannian $\mathrm{G}_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ (represented by the orthogonal projection onto these subspaces) is a submanifold of dimension $k(n-k)$.*

Exercise 9. *Show that $\mathrm{G}_1(\mathbb{R}^3) \subset \mathrm{Sym}(3)$ is diffeomorphic to $\mathbb{R}\mathrm{P}^2$.*

Exercise 10 (Möbius band)**.** *Show that the Möbius band (without boundary)*

$$\mathrm{M} = \left\{ \left((2 + r\cos\tfrac{\varphi}{2})\cos\varphi, (2 + r\cos\tfrac{\varphi}{2})\sin\varphi, r\sin\tfrac{\varphi}{2} \right) \mid r \in (-\tfrac{1}{2}, \tfrac{1}{2}),\ \varphi \in \mathbb{R} \right\}$$

is a submanifold of $\mathbb{R}^3$. Show further that for each point $p \in \mathbb{R}\mathrm{P}^2$ the open set $\mathbb{R}\mathrm{P}^2 \setminus \{p\} \subset \mathbb{R}\mathrm{P}^2$ is diffeomorphic to M.

2. Tangent Vectors

Let M be an n-dimensional smooth manifold. We will define for each $p \in \mathrm{M}$ an n-dimensional vector space $\mathrm{T}_p\mathrm{M}$, the *tangent space of* M *at* p.

Definition 13 (Tangent space)**.** *Let M be a smooth n-manifold and $p \in \mathrm{M}$. A tangent vector X at p is then a linear map*

$$X\colon \mathscr{C}^\infty(\mathrm{M}) \to \mathbb{R}, \quad f \mapsto Xf$$

such that there is a smooth curve $\gamma\colon(-\varepsilon,\varepsilon)\to \mathrm{M}$ *with* $\gamma(0)=p$ *and*

$$Xf=(f\circ\gamma)'(0).$$

The tangent space is then the set of all tangent vectors $\mathrm{T}_p\mathrm{M}:=\{X\mid X \text{ tangent vector at } p\}$.

Let $\varphi=(x_1,\dots,x_n)$ be a chart defined on $U\ni p$. Let $\tilde f=f\circ\varphi^{-1}$, $\tilde\gamma=\varphi\circ\gamma$ and $\tilde p=\varphi(p)$. Then

$$Xf=(f\circ\gamma)'(0)=(\tilde f\circ\tilde\gamma)'(0)=\big(\partial_1\tilde f(\tilde p),\dots,\partial_n\tilde f(\tilde p)\big)\begin{pmatrix}\tilde\gamma_1'(0)\\ \vdots\\ \tilde\gamma_n'(0)\end{pmatrix}.$$

So tangent vectors can be parametrized by n numbers $\alpha_i=\tilde\gamma_i'(0)$:

$$Xf=\alpha_1\partial_1\tilde f(\tilde p)+\cdots+\alpha_n\partial_n\tilde f(\tilde p).$$

Exercise 11. *Within the setup above, show that to each vector* $\alpha\in\mathbb{R}^n$, *there exists a curve* $\gamma\colon(-\varepsilon,\varepsilon)\to\mathrm{M}$ *such that* $\gamma(0)=p$ *and* $(f\circ\gamma)'(0)=\alpha_1\partial_1\tilde f(\tilde p)+\cdots+\alpha_n\partial_n\tilde f(\tilde p)$.

Definition 14 (Coordinate frame)**.** *If* $\varphi=(x_1,\dots,x_n)$ *is a chart at* $p\in\mathrm{M}$, $f\in\mathscr{C}^\infty(\mathrm{M})$. *Then*

$$\frac{\partial}{\partial x_i}\Big|_p f:=\partial_i(f\circ\varphi^{-1})(\varphi(p)),\quad i=1,\dots,n.$$

Interlude: How to construct $\mathscr{C}^\infty$ functions on the whole of M? Toolbox: $f\colon\mathbb{R}\to\mathbb{R}$ with

$$f(x)=\begin{cases}0 & \text{for } x\le 0,\\ e^{-1/x} & \text{for } x>0.\end{cases}$$

is $\mathscr{C}^\infty$ and so is then $g(x)=f(1-x^2)$ and $h(x)=\int_0^x g$. From h this we can build a smooth function $\hat h\colon\mathbb{R}\to[0,1]$ with $\hat h(x)=1$ for $x\in[-\frac14,\frac14]$ and $\hat h(x)=0$ for $x\in\mathbb{R}\setminus(-1,1)$. Then we can define a smooth function $\tilde h\colon\mathbb{R}^n\to\mathbb{R}$ by $\tilde h(x)=\hat h(x_1^2+\cdots x_n^2)$ which vanishes outside the unit ball and is constant $=1$ inside the ball of radius $\frac12$.

Theorem 3. *Let* M *be a smooth* n*-manifold,* $p\in\mathrm{M}$ *and* (U,φ) *a chart with* $U\ni p$. *Let* $a_1,\dots,a_n\in\mathbb{R}$. *Then there is* $f\in\mathscr{C}^\infty(\mathrm{M})$ *such that*

$$\frac{\partial}{\partial x_i}\Big|_p f=a_i,\quad i=1,\dots,n.$$

Proof. We define $\tilde g\colon\mathbb{R}^n\to\mathbb{R}$, $\tilde g(x)=\tilde h(\lambda(x-\varphi(p)))$ with λ such that $\tilde g(x)=0$ for all $x\notin\varphi(U)$. Then let $\tilde f\colon\mathbb{R}^n\to\mathbb{R}$, $\tilde f(x):=\tilde g(x)(a_1x_1+a_2x_2+\cdots+a_nx_n)$. Then $f\colon\mathrm{M}\to\mathbb{R}$ given by

$$f(q)=\begin{cases}\tilde f(\varphi(q)) & \text{for } q\in U,\\ 0 & \text{for } q\notin U\end{cases}$$

is such a function. ∎

Corollary 1. $\frac{\partial}{\partial x_1}\Big|_p,\dots,\frac{\partial}{\partial x_n}\Big|_p$ *are linearly independent.*

Corollary 2. $\mathrm{T}_p\mathrm{M}\subset\mathscr{C}^\infty(\mathrm{M})^*$ *is an* n*-dimensional linear subspace.*

Proof. Follows from the last corollary and from Exercise 11, which shows that $\mathrm{T}_p\mathrm{M}$ is a subspace spanned by $\frac{\partial}{\partial x_1}\Big|_p,\dots,\frac{\partial}{\partial x_n}\Big|_p$. ∎

Theorem 4 (Transformation of coordinate frames)**.** *If* (U,φ) *and* (V,ψ) *are charts with* $p\in U\cap V$*,* $\varphi|_{U\cap V}=\Phi\circ\psi|_{U\cap V}$*. Then for every* $X\in \mathrm{T}_p\mathrm{M}$*,*

$$X=\sum a_i\,\frac{\partial}{\partial x_i}\Big|_p=\sum b_i\,\frac{\partial}{\partial y_i}\Big|_p,$$

where $\varphi=(x_1,\dots,x_n)$*,* $\psi=(y_1,\dots,y_n)$*, we have*

$$\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}=\Phi'(\psi(p))\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}.$$

Proof. Let $\gamma\colon(-\varepsilon,\varepsilon)\to\mathrm{M}$ such that $Xf=(f\circ\gamma)'(0)$. Let $\tilde\gamma=\varphi\circ\gamma$ and $\hat\gamma=\psi\circ\gamma$, then

$$a=\tilde\gamma'(0),\quad b=\hat\gamma'(0).$$

Let $\Phi\colon\psi(U\cap V)\to\varphi(U\cap V)$ be the coordinate change $\Phi=\varphi\circ\psi^{-1}$. Then

$$\tilde\gamma=\varphi\circ\gamma=\phi\circ\psi\circ\gamma=\Phi\circ\hat\gamma.$$

In particular,

$$a=\tilde\gamma'(0)=(\Phi\circ\tilde\gamma)'(0)=\Phi'(\psi(p))\hat\gamma'(0)=\Phi'(\psi(p))b.$$

□

Definition 15. *Let* M *and* $\tilde{\mathrm{M}}$ *be smooth manifolds,* $f\colon\mathrm{M}\to\tilde{\mathrm{M}}$ *smooth,* $p\in\mathrm{M}$*. Then define a linear map* $d_pf\colon\mathrm{T}_p\mathrm{M}\to\mathrm{T}_{f(p)}\tilde{\mathrm{M}}$ *by setting for* $g\in\mathscr{C}^\infty(\tilde{\mathrm{M}})$ *and* $X\in\mathrm{T}_p\mathrm{M}$

$$d_pf(X)g:=X(g\circ f).$$

Remark 5: $d_pf(X)$ *is really a tangent vector in* $\mathrm{T}_p\tilde{\mathrm{M}}$ *because, if* X *corresponds to a curve* $\gamma\colon(-\varepsilon,\varepsilon)\to\mathrm{M}$ *with* $\gamma(0)=p$ *then*

$$d_pf(X)g=\frac{d}{dt}\Big|_{t=0}(g\circ f)\circ\gamma=\frac{d}{dt}\Big|_{t=0}g\circ\underbrace{(f\circ\gamma)}_{=:\tilde\gamma}=\frac{d}{dt}\Big|_{t=0}g\circ\tilde\gamma.$$

Notation: The tangent vector $X\in\mathrm{T}_p\mathrm{M}$ corresponding to a curve $\gamma\colon(-\varepsilon,\varepsilon)\to\mathrm{M}$ with $\gamma(0)=p$ is denoted by $X=:\gamma'(0)$.

Theorem 5 (Chain rule)**.** *Suppose* $g\colon\mathrm{M}\to\tilde{\mathrm{M}}$*,* $f\colon\tilde{\mathrm{M}}\to\hat{\mathrm{M}}$ *are smooth maps. Then*

$$d_p(f\circ g)=d_{g(p)}f\circ d_pg.$$

Definition 16 (Tangent bundle)**.** $\mathrm{TM}:=\sqcup_{p\in\mathrm{M}}\mathrm{T}_p\mathrm{M}$ *is called the tangent bundle of* M*. The map* $\pi\colon\mathrm{TM}\to\mathrm{M},\mathrm{T}_p\mathrm{M}\ni X\to p$ *is called the projection map. So* $\mathrm{T}_p\mathrm{M}=\pi^{-1}(\{p\})$*.*

Elegant version of the chain rule: If $f\colon\mathrm{M}\to\tilde{\mathrm{M}}$ is smooth, then $df\colon\mathrm{M}\to\tilde{\mathrm{M}}$ where $df(X)=d_{\pi(X)}f(X)$. With this notation,

$$d(f\circ g)=df\circ dg.$$

Theorem 6. *If* $f\colon\mathrm{M}\to\tilde{\mathrm{M}}$ *is a diffeomorphism then for each* $p\in\mathrm{M}$ *the map* $d_pf\colon\mathrm{T}_p\mathrm{M}\to\mathrm{T}_{f(p)}\tilde{\mathrm{M}}$ *is a vector space isomorphism.*

Proof. f is bijective and f^{-1} is smooth, $\mathrm{Id_M}=f^{-1}\circ f$. For all $p\in\mathrm{M}$,

$$\mathrm{Id}_{\mathrm{T}_p\mathrm{M}}=d_p(\mathrm{Id_M})=d_{f(p)}f^{-1}\circ d_pf.$$

So d_pf is invertible. □

Theorem 7 (Manifold version of the inverse function theorem)**.** *Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ be smooth, $p \in \mathrm{M}$ with $d_p f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ invertible. Then there are open neighborhoods $U \subset \mathrm{M}$ of p and $V \subset \tilde{\mathrm{M}}$ of $f(p)$ such that $f|_U : U \to V$ is a diffeomorphism.*

Proof. The theorem is a reformulation of the inverse function theorem. □

Theorem 8 (Manifold version of the implicit function theorem - 'submersion theorem')**.** *Let $f\colon \tilde{\mathrm{M}} \to \hat{\mathrm{M}}$ be a submersion, i.e. for each $p \in \tilde{\mathrm{M}}$ the derivative $d_p f\colon \mathrm{T}_p\tilde{\mathrm{M}} \to \mathrm{T}_{f(p)}\hat{\mathrm{M}}$ is surjective. Let $q = f(p)$ be fixed. Then*

$$\mathrm{M} := f^{-1}(\{q\})$$

is an n-dimensional submanifold of $\tilde{\mathrm{M}}$, where $n = \dim \tilde{\mathrm{M}} - \dim \hat{\mathrm{M}}$.

Proof. Take charts and apply Theorem 1. □

Theorem 9 (Immersion theorem)**.** *Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ be an immersion, i.e. for every $p \in \mathrm{M}$ the differential $d_p f\colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_{f(p)}\mathrm{M}$ is injective. Then for each $p \in \mathrm{M}$ there is an open set $U \subset \mathrm{M}$ with $p \in \mathrm{M}$ such that $f(U)$ is a submanifold of $\tilde{\mathrm{M}}$.*

Proof. Take charts and apply Theorem 1. □

Is there a global version, i.e. without passing to $U \subset \mathrm{M}$? Assuming that f is injective is not enough.

Exercise 12. *Let $f\colon \mathrm{N} \to \mathrm{M}$ be a smooth immersion. Prove: If f is moreover a topological embedding, i.e. its restriction $f\colon \mathrm{N} \to f(\mathrm{N})$ is a homeomorphism between N and $f(\mathrm{N})$ (with its subspace topology), then $f(\mathrm{N})$ is a smooth submanifold of M.*

Exercise 13. *Let M be compact, $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ an injective immersion, then $f(\mathrm{M})$ is a submanifold.*

Exercise 14. *Let $\mathrm{X} := \mathbb{C}^2 \setminus \{0\}$. The complex projective plane is the quotient space $\mathbb{C}\mathrm{P}^1 = X/_\sim$, where the equivalence relation is given by*

$$\psi \sim \tilde{\psi} :\Leftrightarrow \lambda\psi = \tilde{\psi}, \quad \lambda \in \mathbb{C}.$$

Consider $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, then the Hopf fibration is the map

$$\pi\colon \mathbb{S}^3 \to \mathbb{C}\mathrm{P}^1, \quad \psi \mapsto [\psi].$$

Show: For each $p \in \mathbb{C}\mathrm{P}^1$ the fiber $\pi^{-1}(\{p\})$ is a submanifold diffeomorphic to $\mathbb{S}^1$.

3. The tangent bundle as a smooth vector bundle

Let M be a smooth n-manifold, $p \in \mathrm{M}$. The tangent space at p is an n-dimensional subspace of $(\mathscr{C}^\infty(\mathrm{M}))^*$ given by

$$\mathrm{T}_p\mathrm{M} = \{X \in (\mathscr{C}^\infty(\mathrm{M}))^* \mid \exists\gamma\colon (-\varepsilon,\varepsilon) \to \mathrm{M},\, \gamma(0) = p,\, Xf = (f\circ\gamma)'(0) = Xf, \forall f \in \mathscr{C}^\infty(\mathrm{M})\}.$$

The *tangent bundle* is then the set

$$\mathrm{TM} = \bigsqcup_{p\in\mathrm{M}} \mathrm{T}_p\mathrm{M}$$

and comes with a projection $\pi\colon \mathrm{TM} \to \mathrm{M}$, $\mathrm{T}_p\mathrm{M} \ni X \mapsto p \in \mathrm{M}$. The set $\pi^{-1}(\{p\}) = \mathrm{T}_p\mathrm{M}$ is called the *fiber of the tangent bundle at p.*

Goal: We want to make TM into a $2n$-dimensional manifold.

If $\varphi = (x_1, \dots, x_n)$ be a chart of M defined on $U \ni p$. Then we have a basis $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p$ of $\mathrm{T}_p\mathrm{M}$. So there are unique $y_1(X), \dots, y_n(X) \in \mathbb{R}$ such that

$$X = \sum y_i(X)\, \frac{\partial}{\partial x_i}\Big|_p .$$

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be a smooth atlas of M. For each $\alpha \in A$ we get an open set $\hat{U}_\alpha := \pi^{-1}(U_\alpha)$ and a function $y_\alpha\colon \hat{U}_\alpha \to \mathbb{R}^n$ which maps a given vector to the coordinates with respect to the frame defined by φ_α, $y_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,n})$. Now, we define $\hat{\varphi}_\alpha\colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by

$$\hat{\varphi}_\alpha = (\varphi_\alpha \circ \pi, y_\alpha).$$

For any two charts we have a transition map $\phi_{\alpha\beta}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ such that $\varphi_\beta|_{U_\alpha \cap U_\beta} = \phi_{\alpha\beta} \circ \varphi_\alpha|_{U_\alpha \cap U_\beta}$. The chain rule yields:

$$y_\beta(X) = \phi'_{\alpha\beta}(\varphi_\alpha(\pi(X)))y_\alpha(X).$$

Hence we see that $\hat{\varphi}_\beta \circ \hat{\varphi}_\alpha^{-1}$ is a diffeomorphism.

Topology on TM**:**

$$\mathcal{O}_{\mathrm{TM}} := \Big\{W \subset \mathrm{TM} \mid \hat{\varphi}_\alpha(W \cap \hat{U}_\alpha) \in \mathcal{O}_{\mathbb{R}^{2n}} \text{ for all } \alpha \in A\Big\}.$$

Exercise 15. *a) This defines a topology on* TM.
b) With this topology TM *is Hausdorff and 2nd-countable.*
c) All $\hat{\varphi}_\alpha$ are homeomrophisms onto their image.

Because coordinate changes are smooth, this turns TM into a smooth $2n$-dimensional manifold.

Definition 17 (Vector field)**.** *A (smooth) vector field on a manifold* M *is a smooth map $X\colon \mathrm{M} \to \mathrm{TM}$ with $\pi \circ X = \mathrm{Id}_{\mathrm{M}}$, i.e. $X(p) \in \mathrm{T}_p\mathrm{M}$ for all $p \in \mathrm{M}$. Usually we write X_p instead of $X(p)$. If X is a vector field and $f \in \mathscr{C}^\infty(\mathrm{M})$, then $Xf \in \mathscr{C}^\infty(M)$ is given by $(Xf)(p) = X_p f$. Read: "X differentiates f".*

Exercise 16. *Show that each of the following conditions is equivalent to the smoothness of a vector field X as a section $X\colon \mathrm{M} \to \mathrm{TM}$:*

a) For each $f \in \mathscr{C}^\infty(\mathrm{M})$, the function Xf is also smooth.
b) If we write $X|_U =: \sum v_i \frac{\partial}{\partial x_i}$ in a coordinate chart $\varphi = (x_1, \dots, x_n)$ defined on $U \subset \mathrm{M}$, then the components $v_i\colon U \to \mathbb{R}$ are smooth.

Exercise 17. *On $\mathbb{S}^2 = \{x = (x_0, x_1, x_2) \mid \|x\| = 1\} \subset \mathbb{R}^3$ we consider coordinates given by the stereographic projection from the north pole $N = (1, 0, 0)$:*

$$y_1 = \tfrac{x_1}{1-x_0}, \quad y_2 = \tfrac{x_2}{1-x_0}.$$

Let the vector fields X and Y on $\mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2 \tfrac{\partial}{\partial y_1} - y_1 \tfrac{\partial}{\partial y_2}, \qquad Y = y_1 \tfrac{\partial}{\partial y_1} + y_2 \tfrac{\partial}{\partial y_2}.$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S = (-1, 0, 0)$.

Exercise 18. *Prove that the tangent bundle of a product of smooth manifolds is diffeomorphic to the product of the tangent bundles of the manifolds. Deduce that the tangent bundle of a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$.*

4. Vector bundles

Definition 18 (Vector bundle)**.** *A smooth vector bundle of rank k is a triple $(\mathrm{E},\mathrm{M},\pi)$ which consists of smooth manifolds E and M and a smooth map $\pi\colon \mathrm{E}\to\mathrm{M}$ such that for each $p\in\mathrm{M}$ the fiber $\mathrm{E}_p := \pi^{-1}(\{p\})$ has the structure of a k-dimensional vector space and each $p\in\mathrm{M}$ has an open neighborhood $U\subset\mathrm{M}$ such that there exists a diffeomorphism*

$$\phi\colon \pi^{-1}(U)\to U\times\mathbb{R}^k$$

such that $\pi_U\circ\phi=\pi$ and for each $p\in\mathrm{M}$ the restriction $\pi_{\mathbb{R}^k}\circ\phi|_{\mathrm{E}_p}$ is a vector space isomorphism.

Definition 19. *Let E be a smooth vector bundle over M. A section of E is a smooth map $\psi\colon\mathrm{M}\to\mathrm{E}$ such that $\pi\circ\psi=\mathrm{Id}_{\mathrm{M}}$. $\Gamma(\mathrm{E}) := \{\psi\colon \mathrm{M}\to\mathrm{E}\mid \psi$ section of $\mathrm{E}\}$.*

Example: *a) We have seen that the tangent bundle TM of a smooth manifold is a vector bundle of rank $\dim\mathrm{M}$. Its smooth sections were called vector fields.*

b) The product $\mathrm{M}\times\mathbb{R}^k$ is called the trivial bundle of rank k. Its smooth sections can be identified with $\mathbb{R}^k$-valued functions. More precisely, if $\pi_2\colon\mathrm{M}\times\mathbb{R}^k\to\mathbb{R}^k$, then

$$\Gamma(\mathrm{M}\times\mathbb{R}^k)\ni\psi\longleftrightarrow f:=\pi_2\circ\psi\in\mathscr{C}^\infty(\mathrm{M}).$$

From now on we will keep this identification in mind.

4.1. Ways to make new vector bundles out of old ones.

General principle: Any linear algebra operation that given new vector spaces out of given ones can be applied to vector bundles over the same base manifold.

Example: *Let E be a rank k vector bundle over M and F be a rank ℓ vector bundle over M.*

a) Then $\mathrm{E}\oplus\mathrm{F}$ denotes the rank $k+\ell$ vector bundle over M the fibers of which are given by $(\mathrm{E}\oplus\mathrm{F})_p=\mathrm{E}_p\oplus\mathrm{F}_p$.

b) Then $\mathrm{Hom}(\mathrm{E},\mathrm{F})$ denotes the rank $k{\cdot}\ell$ vector bundle over M with fiber given by $\mathrm{Hom}(\mathrm{E},\mathrm{F})_p := \{f\colon\mathrm{E}_p\to\mathrm{F}_p\mid f$ linear$\}$.

c) $\mathrm{E}^*=\mathrm{Hom}(\mathrm{E},\mathrm{M}\times\mathbb{R})$ with fibers $(\mathrm{E}^*)_p=(\mathrm{E}_p)^*$.

Let $\mathrm{E}_1,\dots.\mathrm{E}_r,\mathrm{F}$ be vector bundles over M.

d) Then a there is new vector bundle $\mathrm{E}_1^*\otimes\cdots\otimes\mathrm{E}_r^*\otimes\mathrm{F}$ of rank $\mathrm{rank}\mathrm{E}_1\cdots\mathrm{rank}\mathrm{E}_r\cdot\mathrm{rank}\mathrm{F}$ with fiber at p given by $\mathrm{E}_{1p}^*\otimes\cdots\otimes\mathrm{E}_{rp}^*\otimes\mathrm{F}_p=\{\beta\colon\mathrm{E}_{1p}\times\ldots\times\mathrm{E}_{rp}\to\mathrm{F}_p\mid\beta$ multilinear$\}$.

Exercise 19. *Give an explicit description of the (natural) bundle charts for the bundles (written down as sets) in the previous example.*

Starting from TM:

a) $\mathrm{T^*M} := (\mathrm{TM})^*$ is called the *cotangent bundle*.

b) Bundles of multilinear forms with all the $\mathrm{E}_1,\dots,\mathrm{E}_r,\mathrm{F}$ copies of TM, $\mathrm{T^*M}$ or $\mathrm{M}\times\mathbb{R}$ are called *tensor bundles*. Sections of such bundles are called *tensor fields*.

Example: *We have seen that $\mathrm{G}_k(\mathbb{R}^n)=\{$Orthogonal projections onto k-dim subspaces of $\mathbb{R}^n\}$ is an $(n-k)k$-dimensional submanifold of $\mathrm{Sym}(n)$. Now, we can define the tautological bundle as follows:*

$$\mathrm{E}=\{(P,v)\in\mathrm{G}_k(\mathbb{R}^n)\mid Pv=v\}.$$

W is an open neighborhood of P_V as described in the Grassmannian example. Then for $(P_U,v)\in\mathrm{E}$ define $\phi(P_U,v)\in W\times V\cong W\times\mathbb{R}^k$ by $\phi(P_u,v)=(P_U,P_Vv)$. Check that this defines a local trivialization.

Exercise 20. *Let* $\mathrm{M} \subset \mathbb{R}^k$ *be a smooth submanifold of dimension* n*. Let* $\iota \colon \mathrm{M} \hookrightarrow \mathbb{R}^k$ *denote the inclusion map. Show that the normal bundle* $\mathrm{NM} = \sqcup_{p\in\mathrm{M}}(\mathrm{T}_p\mathrm{M})^\perp \subset \iota^*\mathrm{T}\mathbb{R}^k \cong \mathrm{M}\times\mathbb{R}^k$ *is a smooth rank* $k-n$ *vector bundle over* M*.*

Definition 20 (pullback bundle)**.** *Given a smooth map* $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ *and a vector bundle* $\mathrm{E} \to \tilde{\mathrm{M}}$*. Then the pullback bundle* $f^*\mathrm{E}$ *is defined as the disjoint union of the fibers* $(f^*\mathrm{E})_p = \mathrm{E}_{f(p)}$*, i.e.*

$$f^*\mathrm{E} = \bigsqcup_{p\in\mathrm{M}} \mathrm{E}_{f(p)} \subset \mathrm{M}\times\mathrm{E}.$$

Exercise 21. $f^*\mathrm{E}$ *is a smooth submanifold of* $\mathrm{M}\times\mathrm{E}$*.*

Definition 21 (Vector bundle isomorphism)**.** *Two vector bundles* $\mathrm{E}\to\mathrm{M}, \tilde{\mathrm{E}}\to\mathrm{M}$ *are called isomorphic if there is a bundle isomorphism* $f\colon \mathrm{E}\to\tilde{\mathrm{E}}$*, i.e.* $\tilde{\pi}\circ f = \pi$ *(fibers to fibers) and* $f|_{\mathrm{E}_p} : \mathrm{E}_p \to \tilde{\mathrm{E}}_p$ *is a vector space isomorphism.*

Fact (without proof)**:** Every rank k vector bundle E over M is isomorphic to $f^*\tilde{\mathrm{E}}$, where $\tilde{\mathrm{E}}$ is the tautological bundle over $\mathrm{G}_k(\mathbb{R}^n)$ (some n) and some smooth $f\colon \mathrm{M}\to\mathrm{G}_k(\mathbb{R}^n)$.

Definition 22. *A vector bundle* $\mathrm{E}\to\mathrm{M}$ *of rank* k *is called trivial if it is isomorphic to the trivial bundle* $\mathrm{M}\times\mathbb{R}^k$*.*

Remark 6: *If* $\mathrm{E}\to\mathrm{M}$ *is a vector bundle of rank* k *then, by definition, each point* $p\in\mathrm{M}$ *has an open neighborhood* U *such that the restricted bundle* $\mathrm{E}|_U := \pi^{-1}(U)$ *is trivial, i.e. each bundle is locally trivial.*

Definition 23 (Frame field)**.** *Let* $\mathrm{E}\to M$ *be a rank* k *vector bundle,* $\varphi_1,\ldots,\varphi_k \in \Gamma(\mathrm{E})$*. Then* $(\varphi_1,\ldots,\varphi_k)$ *is called a frame field if for each* $p\in\mathrm{M}$ *the vectors* $\varphi_1(p),\ldots,\varphi_k(p)\in\mathrm{E}_p$ *form a basis.*

Proposition 1. E *is trivial if and only if* E *has a frame field.*

Proof. "$\Rightarrow$": E trivial $\Rightarrow \exists F \in \Gamma\mathrm{Hom}(\mathrm{E},\mathrm{M}\times\mathbb{R}^k)$ such that $F_p\colon \mathrm{E}_p \to \{p\}\times\mathbb{R}^k$ is a vector space isomorphism for each p. Then, for $i=1,\ldots,k$ define $\varphi_i\in\Gamma(\mathrm{E})$ by $\varphi_p = F^{-1}(\{p\}\times e_i)$. "$\Leftarrow$": $(\varphi_1,\ldots,\varphi_k)$ frame field $\rightsquigarrow$ define $F\in\Gamma\mathrm{Hom}(\mathrm{E},\mathrm{M}\times\mathbb{R}^k)$ as the unique map such that $F_p(\varphi_i(p)) = \{p\}\times e_i$ for each $p\in\mathrm{M}$. $\rightsquigarrow F$ is a bundle isomorphism. ☐

From the definition of a vector bundle: Each $p\in\mathrm{M}$ has a neighborhood U such that $\mathrm{E}|_U$ has a frame field.

Theorem 10. *For each* $p\in\mathrm{M}$ *there is an open neighborhood* U *and* $\varphi_1,\ldots,\varphi_k\in\Gamma(\mathrm{E})$ *such that* $\varphi_1|_U,\ldots,\varphi_k|_U$ *is a frame field of* $\mathrm{E}|_U$*.*

Proof. There is an open neighborhood $\tilde{U}$ of p such that $\mathrm{E}|_{\tilde{U}}$ is trivial. Thus there is a frame field $\tilde{\varphi}_1,\ldots,\tilde{\varphi}_k\in\Gamma(\mathrm{E}|_{\tilde{U}})$. There is a subset $U\subset\tilde{U}$, a compact subset C with $U\subset C\subset\tilde{U}$ and a smooth function $f\in\mathscr{C}^\infty(\mathrm{M})$ such that $f|_U\equiv 1$ and $f_{\mathrm{M}\setminus C}\equiv 0$. Then, on $\tilde{U}$, we define

$$\varphi_i(q) = f(q)\tilde{\varphi}_i(q), \quad i=1,\ldots,n,$$

and extend it by the 0-vector field to whole of M, i.e. $\varphi_i(q) = 0\in\mathrm{E}_q$ for $q\in\mathrm{M}\setminus\tilde{U}$. ☐

Example: *A rank* 1 *vector bundle* E *(a line bundle) is trivial* $\Leftrightarrow \exists$ *nowhere vanishing* $\varphi\in\Gamma(\mathrm{E})$*.*

Example: $\mathrm{M}\subset\mathbb{R}^\ell$ *submanifold of dimension* $n \rightsquigarrow$ *rank* $\ell-n$ *vector bundle* NM *(the normal bundle of* M*) is given by* $\mathrm{N}_p\mathrm{M} = (\mathrm{NM}) = (\mathrm{T}_p\mathrm{M})^\perp \subset \mathrm{T}_p\mathbb{R}^\ell = \{p\}\times\mathbb{R}^\ell$*. Fact: The normal bundle of a Moebius band is not trivial.*

Example: *The tangent bundle of* $\mathbb{S}^2$ *is not trivial - a fact known as the hairy ball theorem: Every vector field* $X \in \Gamma(\mathrm{T}\,\mathbb{S}^2)$ *has zeros.*

Exercise 22. *Show that the tangent bundle* $\mathrm{T}\mathbb{S}^3$ *of the round sphere* $\mathbb{S}^3 \subset \mathbb{R}^4$ *is trivial.*
Hint: Show that the vector fields $\varphi_1(x_1,x_2,x_3,x_4) = (-x_2,x_1,x_4,-x_3)$*,* $\varphi_2(x_1,x_2,x_3,x_4) = (x_3,x_4,-x_1,-x_2)$ *and* $\varphi_3(x_1,x_2,x_3,x_4) = (-x_4,x_3,-x_2,x_1)$ *form a frame of* $\mathrm{T}\mathbb{S}^3$*.*

5. Vector fields as operators on functions

Let $X \in \Gamma(\mathrm{TM})$, $f \in \mathscr{C}^\infty(\mathrm{M})$. Then $Xf\colon \mathrm{M} \to \mathbb{R}$, $p \mapsto X_p f$, is smooth. So X can be viewed as a linear map $\mathscr{C}^\infty(\mathrm{M}) \to \mathscr{C}^\infty(\mathrm{M})$,

$$f \mapsto Xf.$$

Theorem 11 (Leibniz's rule)**.** *Let* $f, g \in \mathscr{C}^\infty(\mathrm{M})$*,* $X \in \Gamma(\mathrm{TM})$*, then* $X(fg) = (Xf)g + f(Xg)$*.*

Definition 24 (Lie algebra)**.** *A Lie algebra is a vector space* $\mathfrak{g}$ *together with a skew bilinear map* $[.,.]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *which satisfies the Jacobi identity,*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Theorem 12 (Lie algebra of endomorphisms)**.** *Let* V *be a vector space.* $\mathrm{End}(\mathrm{V})$ *together with the commutator* $[.,.]\colon \mathrm{End}(\mathrm{V})\times\mathrm{End}(\mathrm{V}) \to \mathrm{End}(\mathrm{V})$*,* $[A,B] := AB - BA$ *forms a Lie algebra.*

Proof. Certainly the commutaor is a skew bilinear map. Further,

$$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = {} & A(BC-CB) - (BC-CB)A + B(CA-AC) \\ & - (CA-AC)B + C(AB-BA) - (AB-BA)C,\end{aligned}$$

which is zero since each term appears twice but with opposite sign. □

Theorem 13. *For all* $f,g \in \mathscr{C}^\infty(\mathrm{M})$*,* $X,Y \in \Gamma(\mathrm{M})$*,* $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$*.*

Lemma 1 (Schwarz lemma)**.** *Let* $\varphi = (x_1,\dots,x_n)$ *be a coordinate chart. Then* $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$*.*

Exercise 23. *Prove Schwarz lemma above.*

Thus, if $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$, we get

$$[X,Y] = \sum_{i,j}[a_i\frac{\partial}{\partial x_i}, b_j\frac{\partial}{\partial x_j}] = \sum_{i,j}\Big(a_i\frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x_i}\Big) = \sum_{i,j}\Big(a_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\Big)\frac{\partial}{\partial x_i}.$$

Thus $[X,Y] \in \Gamma(\mathrm{TM})$. In particular, we get the following theorem.

Theorem 14. $\Gamma(\mathrm{TM}) \subset \mathrm{End}(\mathscr{C}^\infty(\mathrm{M}))$ *is a Lie subalgebra.*

Exercise 24. *Calculate the commutator* $[X,Y]$ *of the following vector fields on* $\mathbb{R}^2 \setminus \{0\}$*:*

$$X = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}, \qquad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Write X *and* Y *in polar coordinates* $(r\cos\varphi, r\sin\varphi) \mapsto (r,\varphi)$*.*

Definition 25 (Push forward)**.** *Let* $f\colon \mathrm{M} \to \mathrm{N}$ *be a diffeomorphism and* $X \in \Gamma(\mathrm{TM})$*. The push forward* $f_*X \in \Gamma(\mathrm{TN})$ *of* X *is defined by* $f_*X := df \circ X \circ f^{-1}$*.*

Exercise 25. *Let* $f\colon \mathrm{M} \to \mathrm{N}$ *be a diffeomorphism,* $X,Y \in \Gamma(\mathrm{TM})$*. Show:* $f_*[X,Y] = [f_*X, f_*Y]$*.*

6. Connections on vector bundles

Up to now we did basically *Differential Topology*. Now *Differential Geometry* begins, i.e. we study manifolds with additional ("geometric") structure.

Definition 26 (Connection)**.** *A connection on a vector bundle* $\mathrm{E} \to \mathrm{M}$ *is a bilinear map* $\nabla\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E})$ *such that for all* $f \in \mathscr{C}^\infty(\mathrm{M})$*,* $X \in \Gamma(\mathrm{TM})$*,* $\psi \in \Gamma(\mathrm{E})$*,*

$$\nabla_{fX}\psi = f\nabla_X\psi, \quad \nabla_X f\psi = (Xf)\psi + f\nabla_X\psi.$$

The proof of the following theorem will be postponed until we have established the existence of a so called *partition of unity.*

Theorem 15. *On every vector bundle* E *there is a connection* ∇*.*

Definition 27 (Parallel section)**.** *Let* $\mathrm{E} \to \mathrm{M}$ *be a vector bundle with connection* ∇*. Then* $\psi \in \Gamma(\mathrm{E})$ *is called parallel if* $\nabla_X\psi = 0$ *for all* $X \in \mathrm{TM}$*.*

Let ∇, $\tilde\nabla$ be two connections on E. Define $A\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E})$ by $A_X\psi = \tilde\nabla_X\psi - \nabla_X\psi$. Then A satisfies

$$A_{fX}\psi = \tilde\nabla_{fX}\psi - \nabla_{fX}\psi = fA_X\psi$$

and

$$A_X(f\psi) = \cdots = fA_X\psi.$$

Suppose we have $\omega \in \Gamma\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E})$. Then define $B\colon \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \to \Gamma(\mathrm{E})$ by

$$(B_X\psi)_p = \omega_p(X_p)(\psi_p) \in \mathrm{E}_p.$$

Then

$$B_{fX}\psi = fB_X\psi, \qquad B_X(f\psi) = fB_X\psi.$$

Theorem 16 (Characterization of tensors)**.** *Let* E, F *be vector bundles over* M *and* $A\colon \Gamma(\mathrm{E}) \to \Gamma(\mathrm{F})$ *linear such that for all* $f \in \mathscr{C}^\infty(\mathrm{M})$*,* $\psi \in \Gamma(\mathrm{E})$ *we have*

$$A(f\psi) = fA(\psi).$$

Then there is $\omega \in \Gamma\mathrm{Hom}(\mathrm{E}, \mathrm{F})$ *such that* $(A\psi)_p = \omega_p(\psi_p)$ *for all* $\psi \in \Gamma(\mathrm{E})$*,* $p \in \mathrm{M}$*.*

Proof. Let $p \in \mathrm{M}$, $\tilde\psi \in \mathrm{E}_p$. Want to define ω by saying: Choose $\psi \in \Gamma(\mathrm{E})$ such that $\psi_p = \tilde\psi$. Then define $\omega_p(\tilde\psi) = (A\psi)_p$. Claim: $(A\psi)_p$ depends only on ψ_p, i.e. if $\psi, \hat\psi \in \Gamma(\mathrm{E})$ with $\psi_p = \hat\psi_p$ then $(A\psi)_p = (A\hat\psi)_p$, or in other words: $\psi \in \Gamma(\mathrm{E})$ with $\psi_p = 0$ then $(A\psi)_p = 0$. To check this choose a frame field $(\psi_1, \ldots, \psi_k)$ on some neighborhood and a function $f \in \mathscr{C}^\infty(\mathrm{M})$ such that $f\psi_1, \ldots, f\psi_k$ are globally defined sections and $f \equiv 1$ near p. Let $\psi \in \Gamma(\mathrm{E})$ with $\psi_p = 0$. $\rightsquigarrow$ $\psi|\,U = a_1\psi_1 + \cdots + a_k\psi_k$ with $a_1, \ldots, a_k \in \mathscr{C}^\infty(U)$. Then

$$f^2A\psi = A(f^2\psi) = A((fa_1)(f\psi_1) + \cdots + (fa_k)(f\psi_k)) = (fa_1)A(f\psi_1) + \cdots + (fa_k)A(f\psi_k)).$$

Evaluation at p yields then $(A\psi)_p = 0$. □

Remark 7: *In the following we keep this identification be tensors and tensorial maps in mind and just speak of tensors.*

Thus the considerations above can be summarized by the following theorem.

Theorem 17. *Any two connections* ∇ *and* $\tilde\nabla$ *on a vector bundle* E *over* M *differ by a section of* $\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E})$*:*

$$\tilde\nabla - \nabla \in \Gamma\mathrm{Hom}(\mathrm{TM}, \mathrm{End}\,\mathrm{E}).$$

Exercise 26 (Induced connections)**.** *Let* E_i *and* F *denote vector bundles with connections* ∇^i *and* ∇, *respectively. Show that the equation*

$$(\hat{\nabla}_X T)(Y_1, \ldots, Y_r) = \nabla_X(T(Y_1, \ldots, Y_r)) - \sum_i T(Y_1, \ldots, \nabla^i_X Y_i, \ldots, Y_r)$$

for $T \in \Gamma(\mathrm{E}_1^* \otimes \cdots \otimes \mathrm{E}_r^* \otimes \mathrm{F})$ *and vector fields* $Y_i \in \Gamma(\mathrm{E}_i)$ *defines a connection* $\hat{\nabla}$ *on the bundle of multilinear forms* $\mathrm{E}_1^* \otimes \cdots \otimes \mathrm{E}_r^* \otimes \mathrm{F}$.

Remark 8: *Note that, since an isomorphism* $\rho\colon \mathrm{E} \to \tilde{\mathrm{E}}$ *between vector bundles over* M *maps for each* $p \in \mathrm{M}$ *the fiber of* E_p *linearly to the fiber* $\tilde{\mathrm{E}}_p$, *the map* ρ *can be regarded as a section* $\rho \in \Gamma\mathrm{Hom}(\mathrm{E}, \tilde{\mathrm{E}})$. *If moreover* E *is equipped with a connection* ∇ *and* $\tilde{\mathrm{E}}$ *is equipped with a connection* $\tilde{\nabla}$ *we can speak then of parallel isomorphisms:* ρ *is called parallel if* $\hat{\nabla}\rho = 0$, *where* $\hat{\nabla}$ *is the connection on* $\mathrm{Hom}(\mathrm{E}, \tilde{\mathrm{E}})$ *induced by* ∇ *and* $\tilde{\nabla}$ *(compare Example 26 above).*

Definition 28 (Metric)**.** *Let* $\mathrm{E} \to M$ *be a vector bundle and* $\mathrm{Sym}(\mathrm{E})$ *be the bundle whose fiber at* $p \in \mathrm{M}$ *consists of all symmetric bilinear forms* $\mathrm{E}_p \times \mathrm{E}_p \to \mathbb{R}$. *A metric on* E *is a section* $\langle ., . \rangle$ *of* $\mathrm{Sym}(\mathrm{E})$ *such that* $\langle ., . \rangle_p$ *is a Euclidean inner product for all* $p \in \mathrm{M}$. *A vector bundle together with a metric (a pair* $(\mathrm{E}, \langle ., . \rangle)$*) is called Euclidean vector bundle.*

Definition 29 (Metric connection)**.** *Let* $(\mathrm{E}, \langle ., . \rangle)$ *be a Euclidean vector bundle over* M. *Then a connection* ∇ *is called metric if for all* $\psi, \varphi \in \Gamma(\mathrm{E})$ *and* $X \in \Gamma(\mathrm{TM})$ *we have*

$$X\langle \psi, \varphi \rangle = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle.$$

Exercise 27. *Let* ∇ *be a connection on a direct sum* $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$ *of two vector bundles over* M. *Show that*

$$\nabla = \begin{pmatrix} \nabla^1 & A \\ \tilde{A} & \nabla^2 \end{pmatrix},$$

where $\tilde{A} \in \Omega^1(\mathrm{M}, \mathrm{Hom}(\mathrm{E}_1, \mathrm{E}_2))$, $A \in \Omega^1(\mathrm{M}, \mathrm{Hom}(\mathrm{E}_2, \mathrm{E}_1))$ *and* ∇^i *are connections on the bundles* E_i.

Recall: A rank k vector bundle $\mathrm{E} \to \mathrm{M}$ is called trivial if it is isomorphic to *the* trivial bundle $\mathrm{M} \times \mathbb{R}^k$. We know that

$$\mathrm{E} \textit{ trivial} \Leftrightarrow \exists \varphi_1, \ldots, \varphi_k \in \Gamma(\mathrm{E})\colon \varphi_1(p), \ldots, \varphi_k(p) \textit{ linearly independent for all } p \in \mathrm{M}.$$

The trivial bundle comes with a *trivial connection* $\nabla^{trivial}$: $\Gamma(\mathrm{M} \times \mathbb{R}^k) \ni \psi \leftrightarrow f = \pi_2 \circ \psi \in \mathscr{C}^\infty(\mathrm{M}, \mathbb{R}^k)$, then $\nabla^{trivial}_X \psi \leftrightarrow d_X f = Xf$, $X \in \Gamma(\mathrm{TM})$. More precisely,

$$\nabla^{trivial}_X \psi = (\pi(X), Xf).$$

This clarified in the following the trivial connection often will be denoted just by d.

Every vector bundle E is *locally trivial*, i.e. each point $p \in \mathrm{M}$ has an open neighborhood U such that $\mathrm{E}|_U$ is trivial.

Definition 30 (Isomorphism of vector bundles with connection)**.** *An isomorphism between vector bundles with connection* (E, ∇) *and* $(\tilde{\mathrm{E}}, \tilde{\nabla})$ *is a vector bundle isomorphism* $\rho\colon \mathrm{E} \to \tilde{\mathrm{E}}$, *which is parallel, i.e. for all* $X \in \Gamma(\mathrm{TM})$, $\psi \in \Gamma(\mathrm{E})$,

$$\tilde{\nabla}_X(\rho \circ \psi) = \rho \circ (\nabla_X \psi).$$

Two vector bundles with connection are called isomorphic if there exists an isomorphism between them. A vector bundle with connection (E, ∇) *over* M *is called trivial if it is isomorphic to the trivial bundle* $(\mathrm{M} \times \mathbb{R}^k, d)$.

Remark 9: *Note that* $\psi \in \Gamma(\mathrm{M} \times \mathbb{R}^k)$ *is parallel if* $\pi \circ \psi$ *is locally constant.*

Theorem 18. *A vector bundle* E *with connection is trivial iff there exists a parallel frame field.*

Proof. "$\Rightarrow$": Let $\rho\colon \mathrm{M}\times\mathbb{R}^k \to \mathrm{E}$ be a bundle isomorhism such that $\rho\circ d = \nabla\circ\rho$. Then $\phi_{ip} := \rho(p,e_i)$, $i = 1,\dots,k$, form a parallel frame. "$\Leftarrow$": If we have a parallel frame field $\varphi_i \in \Gamma(\mathrm{E})$, then define $\rho\colon \mathrm{M}\times\mathbb{R}^k \to \mathrm{E}$, $\rho(p,v) := \sum v_i\varphi_i(p)$. It is easily checked that ρ is the desired isomorphism. □

Definition 31 (Flat vector bundle)**.** *A vector bundle* E *with connection is called flat if it is locally trivial as a vector bundle with connection, i.e. each point* $p\in\mathrm{M}$ *has an open neighborhood* U *such that* $\mathrm{E}|_U$ *(endowed with the connection inherited from* E*) is trivial. In other words: If there is a parallel frame field over* U*.*

Definition 32 (Bundle-valued differential forms)**.** *Let* $\mathrm{E}\to M$ *be a vector bundle. Then for* $\ell > 0$ *an* E*-valued* ℓ*-form* ω *is a section of the bundle* $\Lambda^\ell(\mathrm{M},\mathrm{E})$ *whose fiber at* $p\in\mathrm{M}$ *is the vector space of multilinear maps* $\mathrm{T}_p\mathrm{M}\times\cdots\times\mathrm{T}_p\mathrm{M}\to\mathrm{E}_p$*, which are alternating, i.e. for* $i\neq j$

$$\omega_p(X_1,\dots,X_i,\dots,X_j,\dots,X_\ell) = -\omega_p(X_1,\dots,X_j,\dots,X_i,\dots,X_\ell).$$

Further, define $\Lambda^0(\mathrm{M},\mathrm{E}) := \mathrm{E}$*. Consequently,* $\Omega^0(\mathrm{M},\mathrm{E}) := \Gamma(\mathrm{E})$*.*

Remark 10: *Each* $\omega\in\Omega^\ell(\mathrm{M},\mathrm{E})$ *defines a tensorial map* $\Gamma(\mathrm{TM})^\ell\to\Gamma(\mathrm{E})$ *and vice versa.*

Definition 33 (Exterior derivative)**.** *Let* $\mathrm{E}\to\mathrm{M}$ *be a vector bundle with connection* ∇*. For* $\ell\geq 0$*, define the exterior derivative* $d^\nabla\colon \Omega^\ell(\mathrm{M},\mathrm{E})\to\Omega^{\ell+1}(\mathrm{M},\mathrm{E})$ *as follows:*

$$\begin{aligned} d^\nabla\omega(X_0,\dots,X_\ell) &= \sum_i (-1)^i\nabla_{X_i}(\omega(X_0,\dots,\hat{X}_i,\dots,X_\ell)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_\ell), \quad X_0,\dots,X_\ell\in\Gamma(\mathrm{TM}). \end{aligned}$$

Proof. Actually there are two things to be verified: $d^\nabla\omega$ is tensorial and alternating. First let us check it is tensorial:

$$\begin{aligned} d^\nabla\omega(X_0,\dots,fX_k,\dots,X_\ell) &= \sum_{i<k}(-1)^i\nabla_{X_i}\omega(X_0,\dots,\hat{X}_i,\dots,fX_k,\dots,X_\ell) \\ &\quad + \nabla_{fX_k}\omega(X_0,\dots,\hat{X}_k,\dots,X_\ell) \\ &\quad + \sum_{i>k}(-1)^i\nabla_{X_i}\omega(X_0,\dots,fX_k,\dots,\hat{X}_i,\dots,X_\ell) \\ &\quad + \sum_{i<j,i\neq k,j\neq k}(-1)^{i+j}\omega([X_i,X_j],\dots,\hat{X}_i,\dots,fX_k,\dots,\hat{X}_j,\dots,X_\ell) \\ &\quad + \sum_{i<k}(-1)^{i+k}\omega([X_i,fX_k],\dots,\hat{X}_i,\dots,\hat{X}_k,\dots,X_\ell) \\ &\quad + \sum_{k<i}(-1)^{k+i}\omega([fX_k,fX_i],\dots,\hat{X}_k,\dots,\hat{X}_i,\dots,X_\ell) \\ &= fd^\nabla\omega(X_0,\dots,fX_k,\dots,X_\ell) \\ &\quad + \sum_{i\neq k}(-1)^i(X_if)\omega(X_0,\dots,\hat{X}_i,\dots,X_\ell) \\ &\quad + \sum_{i<k}(-1)^{i+k}\omega((X_if)X_k,\dots,\hat{X}_i,\dots,\hat{X}_k,\dots,X_\ell) \\ &\quad - \sum_{k<i}(-1)^{k+i}\omega((X_if)X_k,\dots,\hat{X}_k,\dots,\hat{X}_i,\dots,X_\ell) \end{aligned}$$

$$= f d^\nabla \omega(X_0, \dots, fX_k, \dots, X_\ell).$$

Next we want to see that $d^\nabla\omega$ is alternating. Since $d^\nabla\omega$ is tensorial we can test this on commuting vector fields, i.e $[X_i, X_j] = 0$. With this we get for $k < m$ that

$$\begin{aligned}
d^\nabla\omega(X_0, \dots, X_m, \dots, X_k, \dots, X_\ell) &= \sum_{i<k} (-1)^i \nabla_{X_i}\omega(X_0, \dots, \hat{X}_i, \dots, X_m, \dots, X_k, \dots, X_\ell) \\
&\quad + (-1)^k \nabla_{X_m}\omega(X_0, \dots, \hat{X}_m, \dots, X_k, \dots, X_\ell) \\
&\quad + \sum_{k<i<m} (-1)^i \nabla_{X_i}\omega(X_0, \dots, X_m, \dots, \hat{X}_i, \dots, X_k, \dots, X_\ell) \\
&\quad + (-1)^m \nabla_{X_k}\omega(X_0, \dots, X_m, \dots, \hat{X}_m, \dots, X_\ell) \\
&\quad + \sum_{i>k} (-1)^i \nabla_{X_i}\omega(X_0, \dots, X_m, \dots, X_k, \dots, \hat{X}_i, \dots, X_\ell) \\
&= -\sum_{i<k} (-1)^i \nabla_{X_i}\omega(X_0, \dots, \hat{X}_i, \dots, X_k, \dots, X_m, \dots, X_\ell) \\
&\quad + (-1)^{k+(m-k-1)} \nabla_{X_m}\omega(X_0, \dots, X_k, \dots, \hat{X}_k, \dots, X_\ell) \\
&\quad - \sum_{k<i<m} (-1)^i \nabla_{X_i}\omega(X_0, \dots, X_k, \dots, \hat{X}_i, \dots, X_m, \dots, X_\ell) \\
&\quad + (-1)^{m+(m-k-1)} \nabla_{X_k}\omega(X_0, \dots, \hat{X}_k, \dots, X_m, \dots, X_\ell) \\
&\quad - \sum_{i>k} (-1)^i \nabla_{X_i}\omega(X_0, \dots, X_k, \dots, X_m, \dots, \hat{X}_i, \dots, X_\ell) \\
&= \sum_{i<k} (-1)^i \nabla_{X_i}\omega(X_0, \dots, \hat{X}_i, \dots, X_\ell) \\
&= -d^\nabla\omega(X_0, \dots, X_k, \dots, X_m, \dots, X_\ell),
\end{aligned}$$

where the second equation follows by successively shifting the vector fields X_m resp. X_k to the right resp. left. $\square$

1-forms: Let $\mathrm{E} \to \mathrm{M}$ be a vector bundle with connection ∇, then $\Lambda^1(\mathrm{M}, \mathrm{E}) = \mathrm{Hom}(\mathrm{TM}, \mathrm{E})$. We have $\Omega^0(\mathrm{M}, \mathrm{E}) = \Gamma(\mathrm{E})$. We obtain a 1-forms by applying d^∇:

$$\Omega^0(\mathrm{M}, \mathrm{E}) \ni \psi \mapsto d^\nabla\psi = \nabla\psi \in \Omega^1(\mathrm{M}, \mathrm{E}).$$

As a special case we have $\mathrm{E} = \mathrm{M} \times \mathbb{R}$. Then $\Gamma(\mathrm{M} \times \mathbb{R}) \leftrightarrow \mathscr{C}^\infty(\mathrm{M})$ and $\Lambda^1(\mathrm{M}, \mathrm{M} \times \mathbb{R}) = \mathrm{Hom}(\mathrm{TM}, \mathrm{M} \times \mathbb{R}) \leftrightarrow \mathrm{Hom}(\mathrm{TM}, \mathbb{R}) = \mathrm{T}^*\mathrm{M}$. So in this case $\Omega^1(\mathrm{M}, \mathrm{M} \times \mathbb{R}) \cong \Gamma(\mathrm{T}^*\mathrm{M}) = \Omega^1(\mathrm{M})$ (*ordinary* 1-forms are basically sections of $\mathrm{T}^*\mathrm{M}$). For $\mathrm{M} = U \subset \mathbb{R}^n$ (open) we have the standard coordinates $x_i \colon U \to \mathbb{R}$ (projection to the i-component) $\rightsquigarrow dx_i \in \Omega^1(\mathrm{M})$. Let $X_i := \frac{\partial}{\partial x_i} \in \Gamma(\mathrm{T}U)$ which as $\mathbb{R}^n$-valued functions is just the canonical basis $X_i = e_i$. Then $X_1, \dots, X_n$ is a frame: We have $dx_i(X_j) = \delta_{ij}$, thus $dx_1, \dots, dx_n$ is the frame of T^*U dual to $X_1, \dots, X_n$. So every 1-form is of the form:

$$\omega = a_1 dx_1 + \dots + a_n dx_n, \quad a_1, \dots, a_n \in \mathscr{C}^\infty(U).$$

If $f \in \mathscr{C}^\infty(U)$, then $X_i f = \frac{\partial f}{\partial x_i}$. With a small computation we get

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n.$$

ℓ-forms: Let $\mathrm{M} \subset \mathbb{R}^n$ be open and consider again $\mathrm{E} = \mathrm{M} \times \mathbb{R}$. Then for $i_1, \dots, i_\ell$ define $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Omega^\ell(\mathrm{M})$ by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}(X_1, \dots, X_\ell) := \det \begin{pmatrix} dx_{i_1}(X_1) & \cdots & dx_{i_1}(X_\ell) \\ \vdots & \ddots & \vdots \\ dx_{i_\ell}(X_1) & \cdots & dx_{i_\ell}(X_\ell) \end{pmatrix}.$$

Note: If $i_\alpha = i_\beta$ for $\alpha \neq \beta$, then $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} = 0$. If $\sigma\colon \{1, \dots, \ell\} \to \{1, \dots, \ell\}$ is a permutation, we have

$$dx_{i_{\sigma_1}} \wedge \cdots \wedge dx_{i_{\sigma_\ell}} = \operatorname{sign} \sigma \; dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Theorem 19. *Let $U \subset \mathbb{R}^n$ be open. The ℓ-forms $dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$ for $1 \leq i_1 < \cdots < i_\ell \leq n$ are a frame field for $\Lambda^\ell(U)$, i.e. each $\omega \in \Omega^\ell(U)$ can be uniquely written as*

$$\omega = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} a_{i_1 \cdots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$$

with $a_{i_1 \cdots i_\ell} \in \mathscr{C}^\infty(U)$. In fact,

$$a_{i_1 \cdots i_\ell} = \omega\Big(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_\ell}}\Big).$$

Proof. For uniqueness note that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}\Big(\frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_\ell}}\Big) = \det \begin{pmatrix} \delta_{i_1 j_1} & \cdots & \delta_{i_1 j_\ell} \\ \vdots & \ddots & \vdots \\ \delta_{i_\ell j_1} & \cdots & \delta_{i_\ell j_\ell} \end{pmatrix} = \begin{cases} 1 & \textit{if } \{i_1, \dots, i_\ell\} = \{j_1, \dots, j_\ell\}, \\ 0 & \textit{else.} \end{cases}$$

Existence we leave as an exercise. □

Theorem 20. *Let $U \subset \mathbb{R}^n$ be open and $\omega = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} a_{i_1 \cdots i_\ell} \, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Omega^\ell(U)$, then*

$$d\omega = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \sum_{i=1}^n \frac{\partial a_{i_1 \cdots i_\ell}}{\partial x_i} \, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Proof. By Theorem 19 it is enough to show that for all $1 \leq j_0 < \cdots < j_\ell \leq n$

$$\begin{aligned} d\omega\Big(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}\Big) &= \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \sum_{i=1}^n \frac{\partial a_{i_1 \cdots i_\ell}}{\partial x_i} \, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}\Big(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}\Big) \\ &= \sum_{k=0}^\ell \frac{\partial a_{j_0 \cdots \hat{j}_k \cdots j_\ell}}{\partial x_{j_k}} dx_{j_k} \wedge dx_{j_0} \cdots \wedge \widehat{dx_{j_k}} \cdots \wedge dx_{j_\ell}\Big(\frac{\partial}{\partial x_{j_0}}, \dots, \frac{\partial}{\partial x_{j_\ell}}\Big) \\ &= \sum_{k=0}^\ell (-1)^k \frac{\partial a_{j_0 \cdots \widehat{j_k} \cdots j_\ell}}{\partial x_{j_k}}. \end{aligned}$$

But we also get this sum if we apply the definition and use that $[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_m}] = 0$. □

Example: *Let $\mathrm{M} = U \subset \mathbb{R}^3$ be open. Then every $\sigma \in \Omega^2(\mathrm{M})$ can be uniquely written as*

$$\sigma = a_1 dx_2 \wedge dx_3 + a_2 dx_3 \wedge dx_1 + a_3 dx_1 \wedge dx_2.$$

Let $\sigma = d\omega$ with $\omega = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$. Then

$$d\omega\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial x_i}\omega\Big(\frac{\partial}{\partial x_j}\Big) - \frac{\partial}{\partial x_j}\omega\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}.$$

Thus we get that $a = \operatorname{curl}(v)$.

The proofs of Theorem 19 and Theorem 20 directly carry over to bundle-valued forms.

Theorem 21. *Let $U \subset \mathbb{R}^n$ be open and $\mathrm{E} \to U$ be a vector bundle with connection ∇. Then $\omega \in \Omega^\ell(U, \mathrm{E})$ can be uniquely written as*

$$\omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \psi_{i_1 \cdots i_\ell}\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}, \quad \psi_{i_1 \cdots i_\ell} \in \Gamma(\mathrm{E}).$$

Moreover,

$$d^\nabla \omega = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \sum_{i=1}^{n} \big(\nabla_{\frac{\partial}{\partial x_i}} \psi_{i_1 \cdots i_\ell}\big)\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Exercise 28. *Let $\mathrm{M} = \mathbb{R}^2$. Let $J \in \Gamma(\mathrm{End}\mathrm{TM})$ be the 90° rotation and $\det \in \Omega^2(\mathrm{M})$ denote the determinant. Define $*\colon \Omega^1(\mathrm{M}) \to \Omega^1(\mathrm{M})$ by $*\omega(X) = -\omega(JX)$. Show that*

a) for all $f \in \mathscr{C}^\infty(\mathrm{M})$, $d * df = (\Delta f) \det$, where $\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$,
b) $\omega \in \Omega^1(\mathrm{M})$ is closed (i.e. $d\omega = 0$), if and only if ω is exact (i.e. $\omega = df$).

7. Wedge product

Let $\mathrm{U}, \mathrm{V}, \mathrm{W}$ be vector bundles over M. Let $\omega \in \Omega^k(\mathrm{M}, \mathrm{U})$, $\eta \in \Omega^\ell(\mathrm{M}, \mathrm{V})$. We want to define $\omega \wedge \eta \in \Omega^{k+\ell}(\mathrm{M}, \mathrm{W})$. Therefore we need a multiplication $*\colon \mathrm{U}_p \times \mathrm{V}_p \to \mathrm{W}_p$ bilinear such that for $\psi \in \Gamma(\mathrm{U}), \phi \in \Gamma(\mathrm{V})$ such that $\psi * \phi\colon p \mapsto \psi_p *_p \phi_p$ is smooth, i.e. $\psi * \phi \in \Gamma(\mathrm{W})$. In short,

$$* \in \Gamma(\mathrm{U}^* \otimes \mathrm{V}^* \otimes \mathrm{W}).$$

Example: *a) Most standard case:* $\mathrm{U} = \mathrm{M} \times \mathbb{R} = \mathrm{V}$, $*$ *ordinary multiplication in* $\mathbb{R}$.
b) Also useful: $\mathrm{U} = \mathrm{M} \times \mathbb{R}^{k \times \ell}$, $\mathrm{V} = \mathrm{M} \times \mathbb{R}^{\ell \times m}$, $\mathrm{W} = \mathrm{M} \times \mathbb{R}^{k \times m}$, $*$ *matrix multiplication.*
c) Another case: $\mathrm{U} = \mathrm{End}(\mathrm{E})$, $\mathrm{V} = \mathrm{W} = \mathrm{E}$, $*$ *evaluation of endomorphisms on vectors, i.e.* $(A * \psi)_p = A_p(\psi_p)$.

Definition 34 (Wedge product)**.** *Let $\mathrm{U}, \mathrm{V}, \mathrm{W}$ be vector bundles over M and $* \in \Gamma(\mathrm{U}^* \otimes \mathrm{V}^* \otimes \mathrm{W})$. For two forms $\omega \in \Omega^k(\mathrm{M}, \mathrm{U})$ and $\eta \in \Omega^\ell(\mathrm{M}, \mathrm{V})$ the wedge product $\omega \wedge \eta \in \Omega^{k+\ell}(\mathrm{M}, \mathrm{W})$ is then defined as follows*

$$\omega \wedge \eta(X_1, \ldots, X_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn} \sigma\, \omega(X_{\sigma_1}, \ldots, X_{\sigma_k}) * \eta(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+\ell}}).$$

Example (Wedge product of 1-forms)**:** *For $\omega, \eta \in \Omega^1(\mathrm{M})$ we have $\omega(X)\eta(Y) - \omega(Y)\eta(X)$.*

Theorem 22. *Let $\mathrm{U}, \mathrm{V}, \mathrm{W}$ be vector bundles over M, $* \in \Gamma(\mathrm{U}^* \otimes \mathrm{V}^* \otimes \mathrm{W})$, $\tilde{*} \in \Gamma(\mathrm{V}^* \otimes \mathrm{U}^* \otimes \mathrm{W})$ such that $\psi * \phi = \phi \,\tilde{*}\, \psi$ for all $\psi \in \Gamma(\mathrm{U})$ and $\phi \in \Gamma(\mathrm{V})$, then for $\omega \in \Omega^k(\mathrm{M}, \mathrm{U})$, $\eta \in \Omega^\ell(\mathrm{M}, \mathrm{V})$ we have*

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

Proof. The permutation $\rho\colon \{1, \ldots, k+\ell\} \to \{1, \ldots, k+\ell\}$ with $(1, \ldots, k, k+1, \ldots, k+\ell) \mapsto (k+1, \ldots, k+\ell, 1, \ldots, k)$ needs $k\ell$ transpositions, i.e. $\operatorname{sgn} \rho = (-1)^{k\ell}$. Thus

$$\begin{aligned}
\omega \wedge \eta(X_1, \ldots, X_{k+\ell}) &= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn} \sigma\, \omega(X_{\sigma_1}, \ldots, X_{\sigma_k}) * \eta(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+\ell}}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn} \sigma\, \eta(X_{\sigma_{k+1}}, \ldots, X_{\sigma_{k+\ell}}) \,\tilde{*}\, \omega(X_{\sigma_1}, \ldots, X_{\sigma_k}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\, (\sigma \circ \rho)\, \eta(X_{\sigma_{\rho_{k+1}}}, \ldots, X_{\sigma_{\rho_{k+\ell}}}) \,\tilde{*}\, \omega(X_{\sigma_{\rho_1}}, \ldots, X_{\sigma_{\rho_k}})
\end{aligned}$$

$$= \frac{(-1)^{k\ell}}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}\sigma\, \eta(X_{\sigma_1}, \dots, X_{\sigma_k}) \,\tilde{*}\, \omega(X_{\sigma_{k+1}}, \dots, X_{\sigma_{k+\ell}})$$
$$= (-1)^{k\ell}\, \eta \wedge \omega(X_1, \dots, X_{k+\ell}).$$

□

Remark 11: *In particular the above theorem holds for symmetric tensors* $* \in \Gamma(\mathrm{U}^* \otimes \mathrm{U}^* \otimes \mathrm{V})$.

Theorem 23. *Let* $\mathrm{E}_1, \dots, \mathrm{E}_6$ *be vector bundles over* M. *Suppose that* $* \in \Gamma(\mathrm{E}_1^* \otimes \mathrm{E}_2^* \otimes \mathrm{E}_4^*)$, $\tilde{*} \in \Gamma(\mathrm{E}_4^* \otimes \mathrm{E}_3^* \otimes \mathrm{E}_5)$, $\mathring{*} \in \Gamma(\mathrm{E}_1 \otimes \mathrm{E}_6 \otimes \mathrm{E}_5)$ *and* $\hat{*} \in \Gamma(\mathrm{E}_2^* \otimes \mathrm{E}_3^* \otimes \mathrm{E}_6)$ *be associative, i.e.*

$$(\psi_1 * \psi_2) \,\tilde{*}\, \psi_3 = \psi_1 \,\mathring{*}\, (\psi_2 \,\hat{*}\, \psi_3), \text{ for all } \psi_1 \in \Gamma(\mathrm{E}_1), \psi_1 \in \Gamma(\mathrm{E}_2), \psi_1 \in \Gamma(\mathrm{E}_3).$$

Then for $\omega_1 \in \Omega^{k_1}(\mathrm{M}, \mathrm{E}_1)$, $\omega_2 \in \Omega^{k_2}(\mathrm{M}, \mathrm{E}_2)$ *and* $\omega_3 \in \Omega^{k_3}(\mathrm{M}, \mathrm{E}_3)$ *we have*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

Proof. To simplify notation: $\mathrm{E}_1 = \cdots = \mathrm{E}_6 = \mathrm{M} \times \mathbb{R}$ with ordinary multiplication of real numbers. $\omega_1 = \alpha$, $\omega_2 = \beta$, $\omega_3 = \gamma$, $k_1 = k$, $k_2 = \ell$, $k_3 = m$.

$$\alpha \wedge (\beta \wedge \gamma)(X_1, \dots, X_{k+\ell+m}) = \frac{1}{k!(\ell+m)!} \sum_{\sigma \in S_{k+\ell+m}} \operatorname{sgn}\sigma\, \alpha(X_{\sigma_1}, \dots, X_{\sigma_k})$$
$$\cdot \frac{1}{m!} \sum_{\rho \in S_{\ell+m}} \operatorname{sgn}\rho\, \beta(X_{\sigma_{k+\rho_1}}, \dots, X_{\sigma_{k+\rho_\ell}}) \gamma(X_{\sigma_{k+\rho_{\ell+1}}}, \dots, X_{\sigma_{k+\rho_{\ell+m}}})$$

Observe: Fix $\sigma_1, \dots, \sigma_k$. Then $\sigma_{k+1}, \dots, \sigma_{k+\ell+m}$ already account for all possible permutations of the remaining indices. In effect we get the same term $(\ell+m)!$ (number of elements in $S_{\ell+m}$) many times. So:

$$\alpha \wedge (\beta \wedge \gamma)(X_1, \dots, X_{k+\ell+m}) = \frac{1}{k!\ell!m!} \sum_{\sigma \in S_{k+\ell+m}} \operatorname{sgn}\sigma\, \alpha(X_{\sigma_1}, \dots, X_{\sigma_k})$$
$$\cdot \beta(X_{\sigma_{k+1}}, \dots, X_{\sigma_{k+\ell}}) \gamma(X_{\sigma_{k+\ell+1}}, \dots, X_{\sigma_{k+\ell+m}}).$$

Calculation of $(\alpha \wedge \beta) \wedge \gamma$ gives the same result. □

Important special case: On a chart neighborhood (U, φ) of M with $\varphi = (x_1, \dots, dx_n)$ we have

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(Y_1, \dots, Y_k) = \sum_{\sigma \in S_k} \operatorname{sgn}\sigma\, dx_{i_1}(Y_{\sigma_1}) \cdots dx_{i_k}(Y_{\sigma_k}) = \det(dx_{i_j}(Y_k))_{j,k},$$

as was defined previously. In particular, for a bundle-valued form $\omega \in \Omega^\ell(\mathrm{M}, \mathrm{E})$ we obtain with Theorem 21 that

$$\omega|_U = \sum_{1 \le i_1 < \cdots < i_\ell \le n} \psi_{i_1 \cdots i_\ell}\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}, \quad \psi_{i_1 \cdots i_\ell} \in \Gamma(\mathrm{E}|_U),$$

and

$$(d^\nabla \omega)\Big|_U = \sum_{1 \le i_1 < \cdots < i_\ell \le n} d^\nabla \psi_{i_1 \cdots i_\ell} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}.$$

Theorem 24. *Let* $\mathrm{E}_1, \mathrm{E}_2$ *and* E_3 *be vector bundles over* M *with connections* ∇^1, ∇^2 *and* ∇^3, *respectively. Let* $* \in \Gamma(\mathrm{E}_1^* \otimes \mathrm{E}_2^* \otimes \mathrm{E}_3)$ *be parallel, i.e.* $\nabla^3(\psi * \varphi) = (\nabla^1 \psi) * \varphi + \psi * (\nabla^2 \varphi)$ *for all* $\psi \in \Gamma(\mathrm{E}_1)$ *and* $\varphi \in \Gamma(\mathrm{E}_2)$.. *Then, if* $\omega \in \Omega^k(\mathrm{M}, \mathrm{E}_1)$ *and* $\eta \in \Omega^\ell(\mathrm{M}, \mathrm{E}_2)$, *we have*

$$d^{\nabla^3}(\omega \wedge \eta) = (d^{\nabla^1} \omega) \wedge \eta + (-1)^k \omega \wedge (d^{\nabla^2} \eta).$$

Proof. It is enough to show this locally. For $\omega = \psi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = \varphi\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$,

$$\begin{aligned}
d^{\nabla^3}(\omega \wedge \eta) &= d^{\nabla^3}(\psi * \varphi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}) \\
&= d^{\nabla^3}(\psi * \varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= ((d^{\nabla^1}\psi) * \varphi + \psi * d^{\nabla^2}\varphi) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= (d^{\nabla^1}\psi) * \varphi\ \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad + \psi * (d^{\nabla^2}\varphi)\ \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= d^{\nabla^1}\psi \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \varphi\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&\quad + (-1)^k\psi\ \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d^{\nabla^2}\varphi \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= (d^{\nabla^1}\omega) \wedge \eta + (-1)^k\omega \wedge (d^{\nabla^2}\eta).
\end{aligned}$$

Since d^{∇} is $\mathbb{R}$-linear and the wedge product is bilinear the claim follows. ∎

8. Pullback

Motivation: A geodesic in M is a curve γ without acceleration, i.e. $\gamma'' = (\gamma')' = 0$. But what a map is γ'? What is the second prime? $\gamma'(t) \in \mathrm{T}_{\gamma(t)}\mathrm{M}$. Modify γ' slightly $\rightsquigarrow \widehat{\gamma'}(t) = (t, \gamma'(t))$ $\rightsquigarrow \widehat{\gamma'} \in \Gamma(\gamma^*\mathrm{TM})$. Right now $\gamma^*\mathrm{TM}$ is just a vector bundle over $(-\varepsilon, \varepsilon)$. If we had a connection $\widehat{\nabla}$ then we can define

$$\gamma'' = \widehat{\nabla}_{\frac{\partial}{\partial t}}\widehat{\gamma'}.$$

Definition 35 (Pullback of forms). *Let $\omega \in \Omega^k(\tilde{\mathrm{M}}, \mathrm{E})$. Then define $f^*\omega \in \Omega^k(\mathrm{M}, f^*E)$ by*

$$(f^*\omega)(X_1, \ldots, X_k) := (p, \omega(df(X_1), \ldots, df(X_k)))$$

for all $p \in \mathrm{M}$, $X_1, \ldots, X_k \in \mathrm{T}_p\mathrm{M}$. For $\psi \in \Omega^0(\tilde{\mathrm{M}}, \mathrm{E})$ we have $f^*\psi = (\mathrm{Id}, \psi \circ f)$.

For ordinary k-forms $\omega \in \Omega^k(\tilde{\mathrm{M}}) \cong \Omega^k(\tilde{\mathrm{M}}, \tilde{\mathrm{M}} \times \mathbb{R})$: $(f^*\omega)(X_1, \ldots, X_k) = \omega(df(X_1), \ldots, df(X_k))$.

Let $\mathrm{E} \to \tilde{\mathrm{M}}$ be a vector bundle with connection $\tilde{\nabla}$, $f \colon \mathrm{M} \to \tilde{\mathrm{M}}$.

Theorem 25. *There is a unique connection $\nabla =: f^*\tilde{\nabla}$ on $f^*\mathrm{E}$ such that for all $\psi \in \Gamma(\mathrm{E})$, $X \in \mathrm{T}_p\mathrm{M}$ we have $\nabla_X(f^*\psi) = \big(p, \tilde{\nabla}_{df(X)}\psi\big)$. In other words*

$$(f^*\tilde{\nabla})(f^*\psi) = f^*(\tilde{\nabla}\psi).$$

Proof. For uniqueness we choose a local frame field $\varphi_1, \ldots, \varphi_k$ around $f(p)$ defined on $V \subset \mathrm{N}$ and an open neighborhood $U \subset \mathrm{M}$ of p such that $f(U) \subset V$. Then for any $\psi \in \Gamma((f^*\mathrm{E})|_U)$ there are $g_1, \ldots, g_k \in \mathscr{C}^\infty(U)$ such that $\psi = \sum_j g_j f^*\varphi_j$. If a connection ∇ on $f^*\mathrm{E}$ has the desired property then, for $X \in \mathrm{T}_p\mathrm{M}$,

$$\begin{aligned}
\nabla_X\psi &= \sum_j \big((Xg_j)f^*\varphi_j + g_j\nabla_X(f^*\varphi_j)\big) = \sum_j \big((Xg_j)f^*\varphi_j + g_j(p, \tilde{\nabla}_{df(X)}\varphi_j)\big) \\
&= \sum_j \big((Xg_j)f^*\varphi_j + g_j \sum_k (p, \omega_{jk}(X)\varphi_k)\big) = (p, \sum_j \big((Xg_j)\varphi_j \circ f + g_j \sum_k \omega_{jk}(X)\varphi_k \circ f\big),
\end{aligned}$$

where $\tilde{\nabla}_{df(X)}\varphi_j = \sum_k \omega_{jk}(X)\varphi_k \circ f$, $\omega_{jk} \in \Omega^1(U)$. For existence check that this formula *defines* a connection. ∎

Theorem 26. *Let* $\omega \in \Omega^k(\mathrm{M},\mathrm{U})$, $\eta \in \Omega^\ell(\mathrm{M},\mathrm{V})$ *and* $* \in \Gamma(\mathrm{U}^* \otimes \mathrm{V}^* \otimes \mathrm{W})$. *Then*
$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta.$$

Proof. Trivial. □

Theorem 27. *Let* E *be a vector bundle with connection* ∇ *over* $\tilde{\mathrm{M}}$, $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$, $\omega \in \Omega^k(\tilde{\mathrm{M}},\mathrm{E})$. *Then*
$$d^{f^*\nabla}(f^*\omega) = f^*(d^\nabla\omega).$$

Proof. Without loss of generality we can assume that $\tilde{\mathrm{M}} \subset \mathbb{R}^n$ is open and that ω is of the form
$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \psi_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$
Then
$$\begin{aligned} f^*\omega &= \sum_{1\le i_1<\cdots<i_k\le n} (f^*\psi_{i_1\cdots i_k}) f^*dx_{i_1} \wedge \cdots \wedge f^*dx_{i_k},\\ d^\nabla\omega &= \sum_{1\le i_1<\cdots<i_k\le n} \nabla\psi_{i_1\cdots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$
Hence
$$\begin{aligned} f^*d^\nabla\omega &= \sum_{1\le i_1<\cdots<i_k\le n} f^*(\nabla\psi_{i_1\cdots i_k}) \wedge f^*dx_{i_1} \wedge \cdots \wedge f^*dx_{i_k}\\ &= \sum_{1\le i_1<\cdots<i_k\le n} (f^*\nabla f^*\psi_{i_1\cdots i_k}) \wedge dx_{i_1}\circ df \wedge \cdots \wedge dx_{i_k}\circ df\\ &= \sum_{1\le i_1<\cdots<i_k\le n} (d^{f^*\nabla} f^*\psi_{i_1\cdots i_k}) \wedge d(x_{i_1}\circ f) \wedge \cdots \wedge d(x_{i_k}\circ f)\\ &= d^{f^*\nabla}(f^*\omega). \end{aligned}$$
□

Exercise 29. *Consider the polar coordinate map* $f\colon \{(r,\theta) \in \mathbb{R}^2 \mid r > 0\} \to \mathbb{R}^2$ *given by* $f(r,\theta) := (r\cos\theta, r\sin\theta) = (x,y)$. *Show that*
$$f^*(x\,dx + y\,dy) = r\,dr \qquad \text{and} \qquad f^*(x\,dy - y\,dx) = r^2\,d\theta.$$

Theorem 28 (Pullback metric)**.** *Let* $\mathrm{E} \to \tilde{\mathrm{M}}$ *be a Euclidean vector bundle with bundle metric* g *and* $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$. *Then* $f^*\mathrm{E}$ *there is a unique metric* f^*g *such that* $(f^*g)(f^*\psi, f^*\phi) = f^*g(\psi,\phi)$ *and* f^*g *is parallel with respect to the pullback connection* $f^*\nabla$.

Exercise 30. *Prove Theorem 28.*

9. Curvature

Consider the trivial bundle $\mathrm{E} = \mathrm{M}\times\mathbb{R}^k$, then $f \in \mathscr{C}^\infty(\mathrm{M},\mathbb{R}^k) \leftrightarrow \psi \in \Gamma(\mathrm{E})$ by $f \leftrightarrow \psi = (\mathrm{Id}_\mathrm{M}, f)$. On E we have the trivial connection ∇:
$$\psi = (\mathrm{Id}_\mathrm{M}, f) \in \Gamma(\mathrm{E}), \quad X \in \Gamma(\mathrm{TM}) \rightsquigarrow \nabla_X\psi := (\mathrm{Id}_\mathrm{M}, Xf).$$
This ∇ satisfies for all $X, Y \in \Gamma(\mathrm{TM})$, $\psi \in \Gamma(\mathrm{E})$:
$$\nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi = \nabla_{[X,Y]}\psi.$$

Proof. $\nabla_Y\psi = (\mathrm{Id}_\mathrm{M}, Yf)$, $\nabla_X\nabla_Y\psi = (\mathrm{Id}_\mathrm{M}, XYf)$, $\nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi = (\mathrm{Id}_\mathrm{M}, [X,Y]f) = \nabla_{[X,Y]}\psi$. In case $\mathrm{M}\subset\mathbb{R}^n$ open, $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j} \rightsquigarrow [X,Y] = 0$ and the above formula says

$$\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\psi = \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\psi.$$

□

The equation $\nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi = 0$ reflects the fact that for the trivial connection partial derivatives commute. Define a map $\tilde{R}^\nabla\colon \Gamma(\mathrm{TM})\times\Gamma(\mathrm{TM})\times\Gamma(\mathrm{E})\to\Gamma(\mathrm{E})$ by

$$(X,Y,\psi)\mapsto \tilde{R}^\nabla(X,Y)\psi := \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

Theorem 29. *Let* E *be a vector bundle with connection* ∇. *Then for all* $X,Y\in\Gamma(\mathrm{TM})$ *and* $\psi\in\Gamma(\mathrm{E})$ *we have*

$$\tilde{R}^\nabla(X,Y)\psi = d^\nabla d^\nabla\psi(X,Y).$$

Proof. In fact,

$$d^\nabla(d^\nabla\psi)(X,Y) = \nabla_X(d^\nabla\psi(Y)) - \nabla_Y(d^\nabla\psi(X)) - d^\nabla\psi([X,Y]) = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

□

Theorem 30 (Curvature tensor)**.** *Let* ∇ *be a connection on a vector bundle* E *over* M. *The map* $\tilde{R}^\nabla$ *is tensorial in* X,Y *and* ψ. *The corresponding tensor* $R^\nabla\in\Omega^2(\mathrm{M},\mathrm{End}\mathrm{E})$ *such that* $[\tilde{R}^\nabla(X,Y)\psi]_p = R^\nabla(X_p,Y_p)\psi_p$ *is called the curvature tensor of* ∇.

Proof. Tensoriality in X and Y follows from the last theorem. Remains to show that $\tilde{R}^\nabla$ is tensorial in ψ:

$$\begin{aligned}
\tilde{R}^\nabla(X,Y)(f\psi) &= \nabla_X\nabla_Y(f\psi) - \nabla_Y\nabla_X(f\psi) - \nabla_{[X,Y]}(f\psi)\\
&= \nabla_X((Yf)\psi + f\nabla_Y\psi) - \nabla_Y((Xf)\psi + f\nabla_X\psi))\psi - (([X,Y]f)\psi + f\nabla_{[X,Y]}\psi)\\
&= X(Yf)\psi + (Yf)\nabla_X\psi + (Xf)\nabla_Y\psi + f\nabla_X\nabla_Y\psi - Y(Xf)\psi\\
&\quad - (Xf)\nabla_Y\psi - (Yf)\nabla_Y\psi \; - f\nabla_Y\nabla_X\psi - ([X,Y]f)\psi - f\nabla_{[X,Y]}\psi\\
&= f\tilde{R}^\nabla(X,Y)\psi.
\end{aligned}$$

□

Exercise 31. *Let* $\mathrm{E}\to\mathrm{M}$ *be a vector bundle with connection* ∇, $\psi\in\Gamma(E)$ *and* $f\colon\mathrm{N}\to\mathrm{M}$. *Then*

$$(f^*R^\nabla)(f^*\tilde{\psi}) = f^*(R^\nabla\psi) = R^{f^*\nabla}f^*\psi.$$

Lemma 2. *Given* $\hat{X}_1,\dots,\hat{X}_k\in\mathrm{T}_p\mathrm{M}$, *then there are vector fields* $X_1,\dots,X_k\in\Gamma(\mathrm{TM})$ *such that* $X_{1p} = \hat{X}_1,\dots,X_{kp} = \hat{X}_k$ *and there is a neighborhood* $U\ni p$ *such that* $[X_i,X_j]|_U = 0$.

Proof. We have already seen that we can extend coordinate frames to the whole manifold. This yields n vector fields Y_i such that $[Y_i,Y_j]$ vanishes on a neighborhood of p. Since there Y_i form a frame. Then we can build linear combinations of Y_i (constant coefficients) to obtain the desired fields. □

Theorem 31. *Let* $\mathrm{E}\to\mathrm{M}$ *be a vector bundle with connection* ∇. *For each* $\omega\in\Omega^k(\mathrm{M},\mathrm{E})$

$$d^\nabla d^\nabla\omega = R^\nabla\wedge\omega.$$

Proof. Let $p \in \mathrm{M}$, $\hat{X}_1, \dots, \hat{X}_{k+2} \in \mathrm{T}_p\mathrm{M}$. Choose $X_1, \dots, X_{k+2} \in \Gamma(\mathrm{TM})$ such that $X_{ip} = \hat{X}_i$ and near p we have $[X_i, X_j] = 0$, $i, j \in \{1, \dots, k+2\}$. The left side is tensorial, so we can use $X_1, \dots, X_{k+2}$ to evaluate $d^\nabla d^\nabla \omega(\hat{X}_1, \dots, \hat{X}_{k+2})$. Then $i_j \in \{1, \dots, k+2\}$

$$d^\nabla \omega(X_{i_0}, \dots, X_{i_k}) = \sum_{j=0}^{k} (-1)^j \nabla_{X_{i_j}} \omega(X_{i_0}, \dots, \hat{X}_{i_j}, \dots, X_{i_k}).$$

Then

$$\begin{aligned} d^\nabla d^\nabla \omega(X_1, \dots, X_{k+2}) &= \sum_{i<j} (-1)^{i+j} \nabla_{X_i} \nabla_{X_j} \omega(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+2}) \\ &\quad + \sum_{j<i} (-1)^{i+j+1} \nabla_{X_i} \nabla_{X_j} \omega(X_1, \dots, \hat{X}_j, \dots, \hat{X}_i, \dots, X_{k+2}) \\ &= \sum_{i<j} (-1)^{i+j} (\nabla_{X_i} \nabla_{X_j} - \nabla_{X_j} \nabla_{X_i}) \omega(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+2}) \\ &= \sum_{i<j} (-1)^{i+j} R^\nabla(X_i, X_j) \omega(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+2}). \end{aligned}$$

On the other hand

$$R^\nabla \wedge \omega(X_1, \dots, X_{k+2}) = \frac{1}{2 \cdot k!} \sum_{\sigma \in S_{k+2}} \operatorname{sgn} \sigma R^\nabla(X_{\sigma_1}, X_{\sigma_2}) \omega(X_{\sigma_3}, \dots, X_{\sigma_{k+2}}).$$

For $i, j \in \{1, \dots, k+2\}$, $i \neq j$ define

$$A_{\{i,j\}} := \Big\{ \sigma \in S_{k+2} \mid \{\sigma_1, \sigma_2\} = \{i, j\} \Big\}.$$

For $i < j$ define $\sigma^{ij} \in S_{k+2}$ by $\sigma^{ij}_1 = i$ and $\sigma^{ij}_2 = j$, $\sigma^{ij}_3 < \dots < \sigma^{ij}_{k+2}$, i.e.

$$\sigma^{ij} = (i, j, 3, \dots, \hat{i}, \dots, \hat{j}, \dots, k+2).$$

In particular we find that $\operatorname{sgn} \sigma^{ij} = (-1)^{i+j}$. Further

$$A_{\{i,j\}} = \underbrace{\{\sigma^{ij} \circ \rho \mid \rho \in S_{k+2},\ \rho_1 = 1,\ \rho_2 = 2\}}_{=:A^+_{\{i,j\}}} \cup \underbrace{\{\sigma^{ij} \circ \rho \mid \rho \in S_{k+2},\ \rho_1 = 2,\ \rho_2 = 1\}}_{=:A^-_{\{i,j\}}}.$$

Note, $\operatorname{sgn}(\sigma^{ij} \circ \rho) = (-1)^{i+j} \operatorname{sgn} \rho$. With this we get

$$\begin{aligned} R^\nabla \wedge \omega(X_1, \dots, X_{k+2}) &= \frac{1}{2 \cdot k!} \sum_{i<j} \sum_{\sigma \in A_{\{i,j\}}} \operatorname{sgn} \sigma R^\nabla(X_{\sigma_1}, X_{\sigma_2}) \omega(X_{\sigma_3}, \dots, X_{\sigma_{k+2}}) \\ &= \frac{1}{2 \cdot k!} \sum_{i<j} \left(\sum_{\sigma \in A^+_{\{i,j\}}} \operatorname{sgn} \sigma R^\nabla(X_{\sigma_1}, X_{\sigma_2}) \omega(X_{\sigma_3}, \dots, X_{\sigma_{k+2}}) \right. \\ &\qquad \left. + \sum_{\sigma \in A^-_{\{i,j\}}} \operatorname{sgn} \sigma R^\nabla(X_{\sigma_1}, X_{\sigma_2}) \omega(X_{\sigma_3}, \dots, X_{\sigma_{k+2}}) \right) \\ &= \frac{1}{2 \cdot k!} \sum_{i<j} \left(\sum_{\rho \in S_{k+2}, \rho_1 = 1, \rho_2 = 2} (-1)^{i+j} \operatorname{sgn} \rho R^\nabla(X_i, X_j) \omega(X_{\sigma^{ij}_{\rho_3}}, \dots, X_{\sigma^{ij}_{\rho_{k+2}}}) \right. \\ &\qquad \left. + \sum_{\rho \in S_{k+2}, \rho_1 = 2, \rho_2 = 1} (-1)^{i+j} \operatorname{sgn} \rho R^\nabla(X_j, X_i) \omega(X_{\sigma^{ij}_{\rho_3}}, \dots, X_{\sigma^{ij}_{\rho_{k+2}}}) \right) \end{aligned}$$

$$= \frac{1}{2\cdot k!}\sum_{i<j}\left(\sum_{\rho\in S_{k+2},\rho_1=1,\rho_2=2}(-1)^{i+j}\,\mathrm{sgn}\,\rho R^\nabla(X_i,X_j)\mathrm{sgn}\,\rho\,\omega(X_{\sigma^{ij}_3},\dots,X_{\sigma^{ij}_{k+2}})\right.$$

$$\left.+\sum_{\rho\in S_{k+2},\rho_1=2,\rho_2=1}(-1)^{i+j}\mathrm{sgn}\,\rho R^\nabla(X_j,X_i)(-\mathrm{sgn}\,\rho)\,\omega(X_{\sigma^{ij}_{\rho_3}},\dots,X_{\sigma^{ij}_{\rho_{k+2}}})\right)$$

$$= \sum_{i<j}(-1)^{i+j}R^\nabla(X_i,X_j)\omega(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+2})$$

□

Lemma 3. *Let* $\mathrm{E}\to\mathrm{M}$ *be a vector bundle,* $p\in\mathrm{M}$, $\tilde\psi\in\mathrm{E}_p$, $A\in\mathrm{Hom}(\mathrm{T}_p\mathrm{M},\mathrm{E}_p)$. *Then there is* $\psi\in\Gamma(\mathrm{E})$ *such that* $\psi_p=\tilde\psi$ *and* $\nabla_X\psi=A(X)$ *for all* $X\in\mathrm{T}_p\mathrm{M}$.

Proof. Choose a frame field $\varphi_1,\dots,\varphi_k$ of E near p. Then we have near p

$$\nabla_X\varphi_i=\sum_{j=1}^k\alpha_{ij}(X)\varphi_j,\ \textit{for}\ \alpha_{ij}\in\Omega^1(\mathrm{M}),\quad A(X)=\sum_{i=1}^k\beta_i\varphi_{ip}\ \textit{for}\ \beta\in(\mathrm{T}_p\mathrm{M})^*,\quad \hat\psi=\sum_{i=1}^k a_i\varphi_{ip}.$$

Ansatz: $\psi=\sum_i f_i\varphi_i$ near $p\rightsquigarrow$ requirements on f_i. Certainly $f_i(p)=a_i$. Further, for $X\in\mathrm{T}_p\mathrm{M}$,

$$\sum\beta_i(X)\varphi_{ip}=\nabla_X\psi=\sum_i\Big(df_i(X)\varphi_{ip}+f_i(p)\sum_j\alpha_{ij}(X)\varphi_{jp}\Big).$$

With $f_i(p)=a_i$,

$$\beta_i=df_i+\sum_j a_j\alpha_{ji}.$$

Such f_i are easy to find. □

Theorem 32 (Second Bianchi identity)**.** *Let* E *be a vector bundle with connection* ∇. *Then its curvature tensor* $R^\nabla\in\Omega^2(\mathrm{M},\mathrm{End}(\mathrm{E}))$ *satisfies*

$$d^\nabla R^\nabla=0.$$

Proof 1. By the last two lemmas we can just choose $X_0,X_1,X_3\in\Gamma(\mathrm{TM})$ commuting near p and $\psi\in\Gamma(\mathrm{E})$ with $\nabla_X\psi=0$ for all $X\in\mathrm{T}_p\mathrm{M}$. Then near p

$$R^\nabla(X_i,X_j)\psi=\nabla_{X_i}\nabla_{X_j}\psi-\nabla_{X_j}\nabla_{X_i}\psi$$

and thus

$$\begin{aligned}[][d^\nabla R^\nabla(X_0,X_1,X_3)]\psi &= (\nabla_{X_0}R^\nabla(X_1,X_2))\psi+(\nabla_{X_1}R^\nabla(X_2,X_0))\psi+(\nabla_{X_2}R^\nabla(X_0,X_1))\psi\\ &= \nabla_{X_0}\nabla_{X_1}\nabla_{X_2}\psi-\nabla_{X_0}\nabla_{X_2}\nabla_{X_1}\psi+\nabla_{X_1}\nabla_{X_2}\nabla_{X_0}\psi\\ &\quad-\nabla_{X_1}\nabla_{X_0}\nabla_{X_2}\psi+\nabla_{X_2}\nabla_{X_0}\nabla_{X_1}\psi-\nabla_{X_2}\nabla_{X_1}\nabla_{X_0}\psi\\ &= R^\nabla(X_0,X_1)\nabla_{X_2}\psi+R^\nabla(X_2,X_0)\nabla_{X_1}\psi+R^\nabla(X_1,X_2)\nabla_{X_0}\psi,\end{aligned}$$

which vanishes at p. □

Proof 2. We have $(d^\nabla R^\nabla)\psi=d^\nabla(R^\nabla\psi)-R^\nabla\wedge d^\nabla\psi=d^\nabla(d^\nabla d^\nabla\psi)-d^\nabla d^\nabla(d^\nabla\psi)=0$. □

Exercise 32. *Let* $\mathrm{M}=\mathbb{R}^3$. *Determine which of the following forms are closed* ($d\omega=0$) *and which are exact* ($\omega=d\theta$ *for some* θ)*:*

a) $\omega=yz\,dx+xz\,dy+xy\,dz$,
b) $\omega=x\,dx+x^2y^2\,dy+yz\,dz$,
c) $\omega=2xy^2\,dx\wedge dy+z\,dy\wedge dz$.

If ω is exact, please write down the potential form θ explicitly.

Exercise 33. *Let* $\mathrm{M} = \mathbb{R}^n$. *For* $\xi \in \Gamma(\mathrm{TM})$, *we define* $\omega^\xi \in \Omega^1(\mathrm{M})$ *and* $\star\omega^\xi \in \Omega^{n-1}(\mathrm{M})$ *as follows:*

$$\omega^\xi(X_1) := \langle \xi, X_1\rangle, \quad \star\omega^\xi(X_2,\dots,X_n) := \det(\xi, X_2, \dots, X_n), \quad X_1,\dots,X_n \in \Gamma(\mathrm{TM}).$$

Show the following identities:

$$df = \omega^{\operatorname{grad} f}, \quad d\star\omega^\xi = \operatorname{div}(\xi)\det,$$

and for $n=3$,

$$d\omega^\xi = \star\omega^{\operatorname{rot}\xi}.$$

10. Fundamental theorem for flat vector bundles

Let $\mathrm{E} \to \mathrm{M}$ be a vector bundle with connection ∇. Then

$$\mathrm{E} \textit{ trivial} \Longleftrightarrow \exists \textit{ frame field } \Phi = (\varphi_1,\dots,\varphi_k) \textit{ with } \nabla\varphi_i = 0,\ i = 1,\dots,k$$

and

E *flat* $\Longleftrightarrow$ E *locally trivial, i.e. each point* $p \in \mathrm{M}$ *has a neighborhood* U *such that* $\mathrm{E}|_U$ *is trivial.*

Theorem 33 (Fundamental theorem for flat vector bundles)**.** (E,∇) *is flat* $\Longleftrightarrow R^\nabla = 0$.

Proof. "$\Rightarrow$": Let $(\varphi_1,\dots,\varphi_k)$ be a local parallel frame field. Then we have for $i = 1,\dots,k$

$$R^\nabla(X,Y)\varphi_i = \nabla_X\nabla_Y\varphi_i - \nabla_Y\nabla_X\varphi_i - \nabla_{[X,Y]}\varphi_i = 0.$$

Since R^∇ is tensorial checking $R^\nabla\psi = 0$ for the elements of a basis is enough.

"$\Leftarrow$": Assume that $R^\nabla = 0$. Locally we find for each $p \in \mathrm{M}$ a neighborhood U diffeomorphic to $(-\varepsilon,\varepsilon)^n$ and a frame field $\Phi = (\varphi_1,\dots,\varphi_k)$ on U. Define $\omega \in \Omega^1(U, \mathbb{R}^{k\times k})$ by

$$\nabla\varphi_i = \sum_{j=1}^k \varphi_j\omega_{ji}.$$

With $\nabla\Phi = (\nabla\varphi_1,\dots,\nabla\varphi_k)$, we write

$$\nabla\Phi = \Phi\omega.$$

Similarly, for a map $F\colon U \to \mathrm{Gl}(k,\mathbb{R})$ define a new frame field:

$$\tilde\Phi = \Phi F^{-1}$$

All frame fields on U come from such F. We want to choose F in such a way that $\nabla\tilde\Phi = 0$. So,

$$0 \overset{!}{=} \nabla\tilde\Phi = \nabla(\Phi F^{-1}) = (\nabla\Phi)F^{-1} + \Phi d(F^{-1}) = (\nabla\Phi)F^{-1} - \Phi F^{-1}dF\,F^{-1} = \Phi(\omega - F^{-1}dF)F^{-1},$$

where we used that $d(F^{-1}) = -F^{-1}dF\,F^{-1}$. Thus we have to solve

$$dF = F\omega.$$

The *Maurer-Cartan Lemma* (below) states that such $F\colon U \to \mathrm{Gl}(k,\mathbb{R})$ exists if and only if the *integrability condition* (or *Maurer-Cartan equation*)

$$d\omega + \omega\wedge\omega = 0$$

is satisfied. We need to check that in our case the integrability condition holds: We have

$$\begin{aligned}0 = R^{\nabla}(X,Y)\Phi &= \nabla_X\nabla_Y\Phi - \nabla_Y\nabla_X\Phi - \nabla_{[X,Y]}\Phi\\ &= \nabla_X(\Phi\omega(Y)) - \nabla_Y(\Phi\omega(X)) - \Phi\omega([X,Y])\\ &= \Phi\omega(X)\omega(Y) + \Phi(X\omega(Y)) - \Phi\omega(Y)\omega(X) - \Phi(Y\omega(X)) - \Phi\omega([X,Y])\\ &= \Phi(d\omega + \omega\wedge\omega)(X,Y).\end{aligned}$$

Thus $d\omega + \omega\wedge\omega = 0$. $\square$

Exercise 34. *Let* $\mathrm{M}\subset\mathbb{R}^2$ *be open. On* $\mathrm{E} = \mathrm{M}\times\mathbb{R}^2$ *we define two connections* ∇ *and* $\tilde{\nabla}$ *as follows:*

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy\\ x\,dy & 0\end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx\\ x\,dx & 0\end{pmatrix}.$$

Show that (E,∇) *is not trivial. Further construct an explicit isomorphism between* $(\mathrm{E},\tilde{\nabla})$ *and the trivial bundle* (E, d)*.*

Lemma 4 (Maurer-Cartan)**.** *Let* $U := (-\varepsilon,\varepsilon)^n$*,* $\omega\in\Omega^1(\mathrm{U},\mathbb{R}^{k\times k})$*,* $F_0\in\mathrm{Gl}(k,\mathbb{R})$*. Then*

$$\exists F\colon U\to\mathrm{Gl}(k,\mathbb{R}) : dF = F\omega,\ F(0,\dots,0) = F_0 \Longleftrightarrow d\omega+\omega\wedge\omega = 0.$$

Remark 12: *Note that* $d\omega+\omega\wedge\omega$ *automatically vanishes on* 1*-dimensional domains.*

Proof. "$\Rightarrow$": Let $F\colon U\to\mathrm{Gl}(k,\mathbb{R})$ solve the initial value problem $dF = F\omega$, $F(0,\dots,0) = F_0$. Then $0 = d^2F = d(F\omega) = dF\wedge\omega + Fd\omega = F\omega\wedge\omega + Fd\omega = F(d\omega+\omega\wedge\omega)$. Thus $d\omega+\omega\wedge\omega = 0$. "$\Leftarrow$ (Induction on n)": Let $n=1$. We look for $F\colon(-\varepsilon,\varepsilon)\to\mathrm{Gl}(k,\mathbb{R})$ with $dF = F\omega$, $F(0,\dots,0) = F_0\in\mathrm{Gl}(k,\mathbb{R})$. With $\omega = A\,dx$, this becomes just the linear ODE

$$F' = FA,$$

which is solvable. Only thing still to check that $F(x)\in\mathrm{Gl}(k,\mathbb{R})$ for initial value $F_0\in\mathrm{Gl}(k,\mathbb{R})$. But for a solution F we get $(\det F)' = (\det F)\,\mathrm{tr}A$. Thus if $(\det F)(0) = \det F_0\neq 0$ then $\det F(x)\neq 0$ for all $x\in(-\varepsilon,\varepsilon)$. Now let $n>1$ and suppose that the Maurer-Cartan lemma holds for $n-1$. Write $\omega = A_1\,dx_1+\cdots+A_ndx_n$ with $A_i\colon(-\varepsilon,\varepsilon)^n\to R^{k\times k}$. Then

$$(d\omega+\omega\wedge\omega)(\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j}) = \Big(\sum_\alpha dA_\alpha\wedge dx_\alpha + \sum_{\alpha,\beta}A_\alpha A_\beta dx_\alpha\wedge dx_\beta\Big)(\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j}) = \tfrac{\partial A_j}{\partial x_i} - \tfrac{\partial A_i}{\partial x_j} + A_iA_j - A_jA_i.$$

By induction hypothesis there is $\hat{F}\colon(-\varepsilon,\varepsilon)^{n-1}\to\mathrm{Gl}(k,\mathbb{R})$ with $\frac{\partial\hat{F}}{\partial x_i} = \hat{F}A_i$, $i=1,\dots,n-1$, and $\hat{F}(0) = F_0$. Now we solve for each $(x_1,\dots,x_{n-1})$ the initial value problem

$$\tilde{F}'_{x_1,\dots,x_{n-1}}(x_n) = \tilde{F}_{x_1,\dots,x_{n-1}}(x_n)A_n(x_1,\dots,x_n), \quad \tilde{F}_{x_1,\dots,x_{n-1}}(0) = \hat{F}(x_1,\dots,x_{n-1}).$$

Define $F(x_1,\dots,x_n) := \tilde{F}_{x_1,\dots,x_{n-1}}(x_n)$. By construction $\frac{\partial F}{\partial x_n} = FA_n$ and with $d\omega+\omega\wedge\omega = 0$,

$$\begin{aligned}\tfrac{\partial}{\partial x_n}(\tfrac{\partial F}{\partial x_i} - FA_i) &= \tfrac{\partial}{\partial x_i}\tfrac{\partial F}{\partial x_n} - \tfrac{\partial}{\partial x_n}(FA_i) = \tfrac{\partial}{\partial x_i}(FA_n) - \tfrac{\partial}{\partial x_n}(FA_i)\\ &= \tfrac{\partial F}{\partial x_i}A_n - \tfrac{\partial F}{\partial x_n}A_i + F(\tfrac{\partial A_n}{\partial x_i} - \tfrac{\partial A_i}{\partial x_n})\\ &= \tfrac{\partial F}{\partial x_i}A_n - FA_nA_i + F(A_nA_i - A_iA_n)\\ &= F(\tfrac{\partial F}{\partial x_i} - FA_i)A_n.\end{aligned}$$

Thus $t\mapsto(\frac{\partial F}{\partial x_i} - FA_i)(x_1,\dots,x_{n-1},t)$ solves a linear ODE. Since $\frac{\partial F}{\partial x_i} - FA_i = 0$ on the slice $\{x\in(-\varepsilon,\varepsilon)^n \mid x_n = 0\}$, we conclude $\frac{\partial F}{\partial x_\alpha} - FA_\alpha$ for all $\alpha\in\{1,\dots,n\}$ on whole of $(-\varepsilon,\varepsilon)^n$. $\square$

Exercise 35. *Let* $\mathrm{M} \subset \mathbb{R}$ *be an interval and consider the vector bundle* $\mathrm{E} = \mathrm{M} \times \mathbb{R}^k$, $k \in \mathbb{N}$, *equipped with some connection* ∇. *Show that* (E, ∇) *is trivial. Furthermore, show that any vector bundle with connection over an intervall is trivial.*

11. Affine connections

Definition 36. *A connection* ∇ *on the tangent bundle is called an affine connection.*

Special about the tangent bundle is that there exists a canonical 1-form $\omega \in \Omega^1(\mathrm{M}, \mathrm{TM})$, the *tautological form*, given by

$$\omega(X) := X.$$

Definition 37 (Torsion tensor)**.** *If* ∇ *is an affine connection on* M*, the* TM*-valued* 2*-form* $T^\nabla := d^\nabla \omega$ *is called the torsion tensor of* ∇. ∇ *is called torsion-free if* $T^\nabla = 0$.

Example 1: *Let* $\mathrm{M} \subset \mathbb{R}^n$ *open. Identify* TM *with* $\mathrm{M} \times \mathbb{R}^n$ *by setting* $(p, X)f = d_p f(X)$. *On* $\mathrm{M} \times \mathbb{R}$ *use the trivial connection: All* $X \in \Gamma(\mathrm{M} \times \mathbb{R})$ *are of the form* $X = (\mathrm{Id}, \hat{X})$ *for* $\hat{X} \in \mathscr{C}^\infty(\mathrm{M}, \mathbb{R}^n)$.

$$(\nabla_X Y)_p = (p, d_p \hat{Y}(X)).$$

Remark (engineer notation): $\nabla_X Y = (X \cdot \nabla) Y$, *with* $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})^t$ *and* $X = (x_1, x_2, x_3)$

$$X \cdot \nabla = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

Define a frame field $X_1, \ldots, X_n$ *on* M *of 'constant vector fields'* $X_j = (p, e_j)$. *Then with* ∇ *denoting the trivial connection on* $\mathrm{TM} = \mathrm{M} \times \mathbb{R}^n$ *we have*

$$T^\nabla(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = 0.$$

Theorem 34 (First Bianchi identity)**.** *Let* ∇ *be a torsion-free affine connection on* M*. Then for all* $X, Y, Z \in \Gamma(\mathrm{TM})$ *we have*

$$R^\nabla(X, Y)Z + R^\nabla(Y, Z)X + R^\nabla(Z, X)Y = 0.$$

Proof. For the tautological 1-form $\omega \in \Omega^1(\mathrm{M}, \mathrm{TM})$ and a torsion-free connection we have $0 = d^\nabla d^\nabla \omega(X, Y, Z) = R^\nabla \wedge \omega(X, Y, Z) = R^\nabla(X, Y)Z + R^\nabla(Y, Z)X + R^\nabla(Z, X)Y$. □

Theorem 35. *If* ∇ *is a metric connection on a Euclidean vector bundle* $\mathrm{E} \to \mathrm{M}$ *then we have for all* $X, Y \in \Gamma(\mathrm{TM})$ *and* $\psi, \varphi \in \Gamma(\mathrm{E})$

$$\langle R^\nabla(X, Y)\psi, \varphi \rangle = -\langle \psi, R^\nabla(X, Y)\varphi \rangle,$$

i.e. as a 2*-form* R^∇ *takes values in the skew-adjoint endomorphisms.*

Proof. The proof is straightforward. We have

$$\begin{aligned} 0 &= d^2 \langle \psi, \varphi \rangle \\ &= d\langle d^\nabla \psi, \varphi \rangle + d\langle \psi, d^\nabla \varphi \rangle = \langle d^\nabla d^\nabla \psi, \varphi \rangle - \langle d^\nabla \psi \wedge d^\nabla \varphi \rangle + \langle d^\nabla \psi \wedge d^\nabla \varphi \rangle + \langle \psi, d^\nabla d^\nabla \varphi \rangle \\ &= \langle d^\nabla d^\nabla \psi, \varphi \rangle + \langle \psi, d^\nabla d^\nabla \varphi \rangle. \end{aligned}$$

With $d^\nabla d^\nabla = R^\nabla$ this yields the statement. □

Definition 38 (Riemannian manifold)**.** *A Riemannian manifold is a manifold* M *together with a Riemannian metric, i.e. a metric* $\langle ., . \rangle$ *on* TM.

Theorem 36 (Fundamental theorem of Riemannian geometry)**.** *On a Riemannian manifold there is a unique affine connection ∇ which is both metric and torsion-free. ∇ is called the Levi-Civita connection.*

Proof. Uniqueness: Let ∇ be metric and torsion-free, $X, Y, Z \in \Gamma(\mathrm{TM})$. Then

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle \\ &\quad + \langle Z, \nabla_Y X\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= \langle \nabla_X Y + \nabla_Y X, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle \nabla_Y Z - \nabla_Z Y, X\rangle \\ &= \langle 2\nabla_X Y - [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle [Y,Z], X\rangle. \end{aligned}$$

Hence we obtain the so called *Koszul formula*:

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle Y,[X,Z]\rangle - \langle [Y,Z],X\rangle\Big).$$

So ∇ is unique. Conversely define $\nabla_X Y$ by the Koszul formula (for this to make sense we need to check tensoriality). Then check that this defines a metric torsion-free connection. □

Exercise 36. *Let (M, g) be a Riemannian manifold and $\tilde{g} = e^{2u} g$ for some smooth function $u \colon \mathrm{M} \to \mathbb{R}$. Show that between the corresponding Levi-Civita connections the following relation holds:*

$$\tilde{\nabla}_X Y = \nabla_X Y + du(X)Y + du(Y)X - g(X,Y)\operatorname{grad} u.$$

Here $\operatorname{grad} u \in \Gamma(\mathrm{TM})$ *is the vector field uniquely determined by the condition* $du(X) = g(\operatorname{grad} u, X)$ *for all* $X \in \Gamma(\mathrm{TM})$*.*

Definition 39 (Riemannian curvature tensor)**.** *Let* M *be a Riemannian manifold. The curvature tensor R^{∇} of its Levi-Civita connection ∇ is called the Riemannian curvature tensor.*

Exercise 37. *Let $(\mathrm{M}, \langle .,. \rangle)$ be a 2-dimensional Riemannian manifold, R its curvature tensor. Show that there is a function $K \in \mathscr{C}^\infty(\mathrm{M})$ such that*

$$R(X,Y)Z = K\Big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\Big), \text{ for all } X,Y,Z \in \Gamma(\mathrm{TM}).$$

Exercise 38. *Let $\langle .,. \rangle$ be the Euclidean metric on $\mathbb{R}^n$ and $B := \{x \in \mathbb{R}^n \mid |x|^2 < 1\}$. For $k \in \{-1, 0, 1\}$ define*

$$g_k|_x := \frac{4}{(1 + k|x|^2)^2}\langle .,. \rangle.$$

Show that for the curvature tensors R_k of the Riemannian manifolds (B, g_{-1}), $(\mathbb{R}^n, g_0)$ and $(\mathbb{R}^n, g_1)$ and for every $X, Y \in \mathbb{R}^n$ the following equation holds:

$$g_k(R_k(X,Y)Y, X) = k\Big(g_k(X,X)g_k(Y,Y) - g_k(X,Y)^2\Big).$$

12. Flat Riemannian manifolds

The Maurer-Cartan-Lemma states that if $\mathrm{E} \to \mathrm{M}$ is a vector bundle with connection ∇ such that $R^{\nabla} = 0$ then E is flat, i.e. each $p \in \mathrm{M}$ has a neighborhood U and a frame field $\varphi_1, \ldots, \varphi_k \in \Gamma(\mathrm{E}|_U)$ with $\nabla\varphi_j = 0$, $j = 1, \ldots, k$. In fact if we look at the proof we see that given a basis $\psi_1, \ldots, \psi_k \in \mathrm{E}_p$ the frame $\varphi_1, \ldots, \varphi_k$ can be chosen in such a way that $\varphi_j(p) = \psi_j$, $j = 1, \ldots, k$.

Suppose E is Euclidean with compatible ∇ then choose $\psi_1, \ldots, \psi_k$ to be an orthonormal basis. Then for each $X \in \Gamma(\mathrm{TU})$ we have $X\langle \varphi_i, \varphi_j\rangle = 0$, $i, j = 1, \ldots, k$, i.e. (assuming that U is connected) $\varphi_1, \ldots, \varphi_k$ is an orthonormal frame field: $\langle \varphi_i, \varphi_j\rangle(q) = \delta_{ij}$ for all $q \in U$. We summarize this in the following theorem.

Theorem 37. *Every Euclidean vector bundle with flat connection locally admits an orthonormal parallel frame field.*

Definition 40 (Isometry)**.** *Let* M *and* N *be Riemannian manifolds. Then* $f\colon \mathrm{M}\to\mathrm{N}$ *is called an isometry if for all* $p\in\mathrm{M}$ *the map* $d_pf\colon \mathrm{T}_p\mathrm{M}\to\mathrm{T}_{f(p)}\mathrm{N}$ *is an isometry of Euclidean vector spaces. In other words,* f *is a diffeomorphism such that for all* $p\in\mathrm{M}$*,* $X,Y\in\mathrm{T}_p\mathrm{M}$ *we have*

$$\langle df(X), df(Y)\rangle_{\mathrm{N}} = \langle X,Y\rangle_{\mathrm{M}}.$$

Intiution: n-dimensional Riemannian manifolds are "curved versions of $\mathbb{R}^n$". $\mathbb{R}^n$ = *"flat space"*. The curvature tensor R^∇ measures curvature, i.e. deviation from flatness.

The following theorem states that any Riemannian manifold with curvature $R=0$ is locally isometric to $\mathbb{R}^n$.

Theorem 38. *Let* M *be an* n*-dimensional Riemannian manifold with curvature tensor* $R=0$ *and let* $p\in\mathrm{M}$*. Then there is a neighborhood* $U\subset\mathrm{M}$ *of* p*, an open set* $V\subset\mathbb{R}^n$ *and an isometry* $f\colon U\to V$*.*

Proof. Choose $\tilde U\subset\mathrm{M}$ open, $p\in\tilde U$ then there is a parallel orthonormal frame field $X_1,\dots,X_N\in\Gamma(\mathrm{T}\tilde U)$. Now define $\mathrm{E} := \mathrm{TM}\oplus(\mathrm{M}\times\mathbb{R}) = \mathrm{TM}\oplus\mathbb{R}$. Any $\psi\in\Gamma(\mathrm{E})$ is of the form

$$\psi = \begin{pmatrix} Y \\ g \end{pmatrix}$$

with $Y\in\Gamma(\mathrm{TM})$ and $g\in\mathscr{C}^\infty(\mathrm{M})$. Define a connection $\tilde\nabla$ on E as follows

$$\tilde\nabla_X \begin{pmatrix} Y \\ g \end{pmatrix} := \begin{pmatrix} \nabla_X Y - gX \\ Xg \end{pmatrix}.$$

It is easy to see that $\tilde\nabla$ is a connection. Now

$$\begin{aligned}
R^{\tilde\nabla}(X,Y)\begin{pmatrix} Z \\ g \end{pmatrix} &= \tilde\nabla_X \begin{pmatrix} \nabla_Y Z - gY \\ Yg \end{pmatrix} - \tilde\nabla_Y \begin{pmatrix} \nabla_X Z - gX \\ Xg \end{pmatrix} - \begin{pmatrix} \nabla_{[X,Y]} Z - g[X,Y] \\ [X,Y]g \end{pmatrix} \\
&= \begin{pmatrix} \nabla_X\nabla_Y Z - (Xg)Y - g\nabla_X Y - (Yg)X \\ XYg \end{pmatrix} - \begin{pmatrix} \nabla_Y\nabla_X Z - (Yg)X - g\nabla_Y X - (Xg)Y \\ YXg \end{pmatrix} - \begin{pmatrix} \nabla_{[X,Y]} Z - g[X,Y] \\ [X,Y]g \end{pmatrix} \\
&= \begin{pmatrix} R(X,Y)Z \\ 0 \end{pmatrix} = 0.
\end{aligned}$$

Now choose $\hat U\subset\tilde U$, $p\in\hat U$ and $\psi\in\Gamma(\mathrm{E}|_{\hat U})$ with $\psi_p=(0,1)$, $\tilde\nabla\psi=0$. Then $\psi=(Y,g)$ with $Y=\sum_{j=1}^n f_jX_j$ and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_X Y - gX \\ Xg \end{pmatrix} = \begin{pmatrix} \sum_j df_j(X)X_j - gX \\ Xg \end{pmatrix}.$$

In particular, $g=1$. If we define $f\colon\hat U\to\mathbb{R}^n$ by $f=(f_1,\dots,f_n)$ then

$$\langle df(X), df(Z)\rangle = \sum_j \langle df_j(X), df_j(Z)\rangle = \langle gX, gY\rangle = \langle X,Y\rangle.$$

In particular, d_pf is bijective. The inverse function theorem then yields a neighborhood U of p such that $f|_U : U\to V\subset\mathbb{R}^n$ is a diffeomorphism and hence an isometry. $\square$

Exercise 39. *Let* M *and* $\tilde{\mathrm{M}}$ *be Riemannian manifolds with Levi-Civita connections* ∇ *and* $\tilde\nabla$*, respectively. Let* $f\colon M\to\tilde M$ *be an isometry and* $X,Y\in\Gamma(\mathrm{M})$*. Show that* $f_*\nabla_XY = \tilde\nabla_{f_*X}f_*Y$*.*

Remark 13: *With the last exercise follows that a Riemannian manifold* M *has curvature* $R=0$ *if and only if it is locally isometric to* $\mathbb{R}^n$*.*

Exercise 40. *a) Show that* $\langle X,Y\rangle := \frac{1}{2}\mathrm{trace}(\bar X^tY)$ *defines a Riemannian metric on* $\mathrm{SU}(2)$*.*
b) Show that the left and the right multiplication by a constant g *are isometries.*
c) Show that $\mathrm{SU}(2)$ *and the* 3*-sphere* $\mathbb{S}^3\subset\mathbb{R}^4$ *(with induced metric) are isometric.*
Hint: $\mathrm{SU}(2) = \{\begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \mid a,b\in\mathbb{C}, |a|^2+|b|^2=1\}$.

13. Geodesics

Let M be a Riemannian manifold, ∇ the Levi-Civita connection on TM, $\gamma\colon [a,b] \to \mathrm{M}$, $Y \in \Gamma(\gamma^*\mathrm{TM})$. Then, for $t \in [a,b]$ we have $Y_t \in (\gamma^*\mathrm{TM})_t = \{t\} \times \mathrm{T}_{\gamma(t)}\mathrm{M} \cong \mathrm{T}_{\gamma(t)}\mathrm{M}$. Y is called a *vector field along* γ. Now define $(Y')_t = (\gamma^*\nabla)_{\frac{\partial}{\partial s}\big|_t} Y =: \frac{dY}{ds}(t)$.

Definition 41 (Geodesic)**.** *$\gamma\colon [a,b] \to \mathrm{M}$ is called a geodesic if $\gamma'' = 0$.*

Exercise 41. *Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ and $g\colon \tilde{\mathrm{M}} \to \hat{\mathrm{M}}$ be smooth. Show that $f^*(g^*\mathrm{T}\hat{\mathrm{M}}) \cong (g \circ f)^*\mathrm{T}\hat{\mathrm{M}}$ and*

$$(g \circ f)^*\hat{\nabla} = f^*(g^*\hat{\nabla})$$

for any affine connection $\hat{\nabla}$ on $\hat{\mathrm{M}}$. Show further that, if f is an isometry between Riemannian manifolds, γ is curve in M and $\tilde{\gamma} = f \circ \gamma$, then

$$\tilde{\gamma}'' = df(\gamma'').$$

Exercise 42. *Let M be a Riemannian manifold, $\gamma\colon I \to \mathrm{M}$ be a curve which is parametrized with constant speed, and $f\colon M \to M$ be an isometry which fixes γ, i.e. $f \circ \gamma = \gamma$. Furthermore, let*

$$\ker(\mathrm{id} - d_{\gamma(t)}f) = \mathbb{R}\dot{\gamma}(t), \text{ for all } t.$$

Then γ is a geodesic.

Definition 42 (Variation)**.** *A variation of $\gamma\colon [a,b] \to \mathrm{M}$ is a smooth map $\alpha\colon (-\varepsilon,\varepsilon)\times[a,b] \to \mathrm{M}$ such that $\gamma_0 = \gamma$, where $\gamma_t\colon [a,b] \to \mathrm{M}$ such that $\gamma_t(s) = \alpha(t,s)$. The vector field along γ given by $Y_s := \frac{d}{dt}\Big|_{t=0} \alpha(t,s)$ is called the variational vector field of α.*

Definition 43 (Length and energy of curves)**.** *Let $\gamma\colon [a,b] \to \mathrm{M}$ be a smooth curve. Then*

$$L(\gamma) := \int_a^b |\gamma'| \text{ is called the length of } \gamma,$$
$$E(\gamma) := \tfrac{1}{2}\int_a^b |\gamma'|^2 \text{ is called the energy of } \gamma.$$

Theorem 39. *Let $\gamma\colon [a,b] \to \mathrm{M}$ be a smooth curve. Let $\varphi\colon [c,d] \to [a,b]$ be smooth with $\varphi'(t) > 0$ for all $t \in [c,d]$, $\varphi(c) = a$ and $\varphi(d) = b$. Then*

$$L(\gamma \circ \varphi) = L(\gamma).$$

Proof. $L(\gamma \circ \varphi) = \int_c^d |(\gamma \circ \varphi)'| = \int_c^d |(\gamma' \circ \varphi)|\varphi' = \int_{\varphi(c)}^{\varphi(d)} |\gamma'| = \int_a^b |\gamma'| = L(\gamma)$. □

Theorem 40. *$E(\gamma) \geq \frac{1}{2(b-a)}L(\gamma)^2$ (equality if and only if $|\gamma'|$ is constant).*

Proof. The Cauchy-Schwarz inequality yields $L(\gamma)^2 \leq 2E(\gamma)\int_a^b 1 = 2(b-a)E(\gamma)$. □

Theorem 41. *Let $\gamma\colon [a,b] \to \mathrm{M}$ be a smooth curve such that $\gamma'(t) \neq 0$ for all $t \in [a,b]$. Then there is a smooth function $\varphi\colon [0, L(\gamma)] \to [a,b]$ with $\varphi'(t) > 0$ for all t, $\varphi(0) = a$ and $\varphi(L(\gamma)) = b$ such that $\tilde{\gamma} = \gamma \circ \varphi$ is arclength parametrized, i.e. $|\tilde{\gamma}'| = 1$.*

Proof. If $\varphi' = 1/|\gamma' \circ \varphi|$, then $|\tilde{\gamma}'| = |(\gamma' \circ \varphi)\varphi'| = 1$. Define $\psi\colon [a,b] \to [0, L(\gamma)]$ by $\psi(t) = \int_a^t |\gamma'|$. Then $\psi'(t) > 0$ for all t, $\psi(a) = 0$ and $\psi(b) = L(\gamma)$. Now set $\varphi = \psi^{-1}$. Then $\varphi' = 1/|\gamma' \circ \varphi|$. □

Theorem 42. *Let $\tilde{\mathrm{M}}$ be a manifold with torsion-free connection $\tilde{\nabla}$. Let $f\colon \mathrm{M} \to \tilde{\mathrm{M}}$ and let $\tilde{\nabla} = f^*\nabla$ be the pullback connection on $f^*\mathrm{T}\tilde{\mathrm{M}}$. Then, if $X, Y \in \Gamma(\mathrm{TM})$ we have $df(X), df(Y) \in \Gamma(f^*\mathrm{TM})$ and*

$$\tilde{\nabla}_X df(Y) - \tilde{\nabla}_Y df(X) = df([X,Y]).$$

Proof. Let ω denote the tautological 1-form on $\mathrm{T}\tilde{\mathrm{M}}$. Then $d^{\tilde{\nabla}}\omega = T^{\tilde{\nabla}} = 0$ and $f^*\omega = df$. Thus

$$0 = f^* d^{\tilde{\nabla}}\omega = d^{\nabla} f^*\omega = d^{\nabla} df.$$

Thus $0 = d^{\nabla} df(X,Y) = \nabla_X df(Y) - \nabla_Y df(X) - df([X,Y])$. □

Example 2: *Let $\mathrm{M} \subset \mathbb{R}^n$ be open, $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$. Then $df(X) = \frac{\partial f}{\partial x_i}$ and $df(Y) = \frac{\partial f}{\partial x_j}$. We have $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$. Hence*

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial f}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial f}{\partial x_i}.$$

Theorem 43 (First variational formula for energy)**.** *Suppose $\alpha\colon (-\varepsilon,\varepsilon) \times [a,b] \to \mathrm{M}$ is a variation of $\gamma\colon [a,b] \to \mathrm{M}$ with variational vector field $Y \in \Gamma(\gamma^*\mathrm{TM})$. Then*

$$\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = \langle Y, \gamma'\rangle\big|_a^b - \int_a^b \langle Y, \gamma''\rangle.$$

Proof.

$$\begin{aligned}
\tfrac{d}{dt}\Big|_{t=0} E(\gamma_t) &= \tfrac{d}{dt}\Big|_{t=0} \tfrac{1}{2}\int_a^b |\gamma_t'|^2 = \tfrac{1}{2}\int_a^b \tfrac{d}{dt}\Big|_{t=0} |\tfrac{\partial\alpha}{\partial s}|^2 = \int_a^b \langle (\alpha^*\nabla)_{\frac{\partial}{\partial t}} \tfrac{\partial\alpha}{\partial s}\Big|_{(0,s)}, \tfrac{\partial\alpha}{\partial s}\rangle \\
&= \int_a^b \langle (\alpha^*\nabla)_{\frac{\partial}{\partial s}} \tfrac{\partial\alpha}{\partial t}\Big|_{(0,s)}, \tfrac{\partial\alpha}{\partial s}\rangle = \int_a^b \tfrac{\partial}{\partial s}\Big|_{(0,s)} \langle \tfrac{\partial\alpha}{\partial t}, \tfrac{\partial\alpha}{\partial s}\rangle - \int_a^b \langle \tfrac{\partial\alpha}{\partial t}, (\alpha^*\nabla)_{\frac{\partial}{\partial s}} \tfrac{\partial\alpha}{\partial s}\Big|_{(0,s)}\rangle \\
&= \int_a^b \tfrac{d}{ds}\langle Y, \gamma'\rangle - \int_a^b \langle Y, \gamma''\rangle = \langle Y, \gamma'\rangle\big|_a^b - \int_a^b \langle Y, \gamma''\rangle.
\end{aligned}$$

□

Corollary 3. *If α is a variation of γ with fixed endpoints, i.e. $\alpha(t,a) = \gamma(a)$ and $\alpha(t,b) = \gamma(b)$ for all $t \in (-\varepsilon,\varepsilon)$, and γ is a geodesic, then $\frac{d}{dt}\Big|_{t=0} E(\gamma_t) = 0$.*

Later we will see the converse statement: If γ is a critical point of E, then γ is a geodesic.

Existence of geodesics: Let ∇ be an affine connection on an open submanifold $\mathrm{M} \subset \mathbb{R}^n$. Let $X_i := \frac{\partial}{\partial x_i}$. Then there are functions Γ_{ij}^k, called Christoffel symbols of ∇, such that

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a smooth curve in M. Then $\gamma' = \sum_i \gamma_i'(\gamma^* X_i)$. By definition of $\gamma^*\nabla$,

$$(\gamma^* X_j)' = (\gamma^*\nabla)_{\frac{\partial}{\partial s}} \gamma^* X_j = \nabla_{\gamma'} X_j = \sum_i \gamma_i' \gamma^*(\nabla_{X_i} X_j) = \sum_{i,k} \gamma_i' (\Gamma_{ij}^k \circ \gamma)\gamma^* X_k.$$

Thus γ is a geodesic of ∇ if and only if

$$0 = \gamma'' = \sum_j \Big(\gamma_j'' \gamma^* X_j + \gamma_j' \sum_{i,k} \gamma_i' (\Gamma_{ij}^k \circ \gamma)\gamma^* X_k\Big).$$

Since $\gamma^* X_i$ form a frame field we get n equations:

$$0 = \gamma_k'' + \sum_{i,j} \gamma_i' \gamma_j' \Gamma_{ij}^k \circ \gamma.$$

This is an ordinary differential equation of second order and Picard-Lindelöf assures the existence of solutions.

Theorem 44 (First variational formula for length)**.** *Let* $\gamma\colon [0,L]\to \mathrm{M}$ *be arclength parametrized, i.e.* $|\gamma'|=1$*. Let* $t\to\gamma_t$ *for* $t\in(-\varepsilon,\varepsilon)$ *be a variation of* γ *with variational vector field* Y*. Then*

$$\frac{d}{dt}\Big|_{t=0} L(\gamma_t)=\langle Y,\gamma'\rangle\big|_0^L-\int_0^L\langle Y,\gamma''\rangle.$$

Proof. Almost the same as for the first variational formula for energy. □

Theorem 45. *Let* $\gamma\colon[a,b]\to\mathrm{M}$ *be a geodesic. Then* $|\gamma'|=$ *constant.*

Proof. We have $\langle\gamma',\gamma'\rangle'=2\langle\gamma',\gamma''\rangle=0$. □

Definition 44 (Killing fields)**.** *Suppose* $t\to g_t$ *for* $t\in(-\varepsilon,\varepsilon)$ *is a* 1*-parameter family of isometries of* M*, i.e. each* $g_t\colon\mathrm{M}\to\mathrm{M}$ *is an isometry. Then the vector field* $X\in\Gamma(\mathrm{TM})$*,* $X_p=\frac{d}{dt}\Big|_{t=0}g_t(p)$ *is called a Killing field of* M*.*

Theorem 46. *Let* $X\in\Gamma(\mathrm{TM})$ *be a Killing field and* $\gamma\colon[a,b]\to\mathrm{M}$ *be a geodesic. Then*

$$\langle X,\gamma'\rangle=\textit{constant}.$$

Proof. Let $\gamma_t:=g_t\circ\gamma$. Then $Y_s=X_{\gamma(s)}$ and $L(\gamma_t)=L(\gamma)$ for all t. Thus

$$0=\tfrac{d}{dt}\Big|_{t=0}L(\gamma_t)=\langle X_\gamma,\gamma'\rangle\big|_a^b-\int_a^b\langle X_\gamma,\gamma''\rangle=\langle X_\gamma,\gamma'\rangle\big|_a^b.$$

Thus $\langle X_{\gamma(a)},\gamma'(a)\rangle=\langle X_{\gamma(b)},\gamma'(b)\rangle$. □

Example 3 (Surface of revolution and Clairaut's relation)**:** *If we have a surface of revolution in Euclidean* 3*-space, then the rotations about the axis of revolution are isometries of the surface. This yields a Killing field* X *such that* X *is orthogonal to the axis of revolution and* $|X|=r$*, where* r *denotes the distance to the axis. From the last theorem we know that if* γ *is a geodesic parametrized with unit speed then* $r\cos\alpha=\langle\gamma',X\rangle=c\in\mathbb{R}$*. Thus* $r=c/\cos\alpha$ *and, in particular,* $r\geq c$ *Thus, depending on the constant* c*, geodesics cannot pass arbitrarily thin parts.*

Example 4 (Rigid body motion)**:** *Let* $\mathrm{M}=\mathrm{SO}(3)\subset\mathbb{R}^{3\times3}$*,* $q_1,\dots,q_n\in\mathbb{R}^3$*,* $m_1,\dots,m_n>0$*. Now if* $t\to A(t)$*,* $t\in(-\varepsilon,\varepsilon)$*,* $B=A(0)$*,* $X=A'(0)$*. Then define*

$$\langle X,X\rangle=\tfrac12\sum_{i=1}^n m_i|Xq_i|^2,$$

where $Xq_i=\frac{d}{dt}|_{t=0}(A(t)q_i)$*.* $\langle X,X\rangle$ *is called the kinetic energy at time* 0 *of the rigid body that undergoes the motion* $t\to A(t)$*. The principle of least action then says: When no forces act on the body, it will move according to* $s\to A(s)\in\mathrm{SO}(3)$ *which is a geodesic. For all* $G\in\mathrm{SO}(3)$ *the left multiplication* $A\mapsto GA$ *is an isometry. Suitable families* $t\mapsto G_t$ *with* $G_0=I$ *then yields the conservation of angular momentum. We leave the details as exercise.*

Theorem 47 (Rope construction of spheres)**.** *Given* $p\in\mathrm{M}$*, for* $t\in[0,1]$ *let* $\gamma_t\colon[0,1]\to\mathrm{M}$ *such that* $\gamma_t(0)=p$ *for all* t*. Let* $X(t)\in\mathrm{T}_p\mathrm{M}$ *such that* $X(t)=\gamma_t'(0)$*,* $|X|=v\in\mathbb{R}$*,* $\eta\colon[0,1]\to\mathrm{M}$*,* $\eta(t)=\gamma_t(1)$*. Then for all* t *we have*

$$\langle\eta'(t),\gamma_t'(1)\rangle=0.$$

Proof. Apply the first variational formula to $\gamma = \gamma_t$: Then we have $Y_0 = 0$ and $Y_1 = \eta'$. Since $L(\gamma_t) = \int_0^1 |\gamma'| = \int_0^1 |X(0)| = v$, we have

$$0 = \frac{d}{dt}\Big|_{t=t_0} L(\gamma_t) = \langle \eta'(t_0), \gamma'_{t_0}(1)\rangle - \langle 0, \gamma'_{t_0}(0)\rangle = \langle \eta'(t_0), \gamma'_{t_0}(1)\rangle.$$

□

14. The exponential map

Theorem 48. *For each $p \in \mathrm{M}$ there is a neighborhood $U \subset \mathrm{M}$ and $\varepsilon > 0$ such that for all $X \in \mathrm{T}_q\mathrm{M}$, $q \in U$, with $|X| < \varepsilon$ there is a geodesic $\gamma\colon [0,1] \to \mathrm{M}$ such that $\gamma(0) = q$, $\gamma'(0) = X$.*

Proof. Picard-Lindelöf yields a neighborhood $\tilde{W} \subset \mathrm{TM}$ of $0 \in \mathrm{T}_p\mathrm{M}$ and $\varepsilon_1 > 0$ such that for $X \in \tilde{W}$, $X \in \mathrm{T}_q\mathrm{M}$, there is a geodesic $\gamma\colon [-\varepsilon_1, \varepsilon_1] \to \mathrm{M}$ such that $\gamma(0) = q$ and $\gamma'(0) = X$. Choose $U \subset \mathrm{M}$ open, $\varepsilon_2 > 0$ such that $W := \{X \in \mathrm{T}_q\mathrm{M} \mid q \in U, |X| \leq \varepsilon_2\} \subset \tilde{W}$. Now set $\varepsilon = \varepsilon_1\varepsilon_2$. Let $q \in U$, $X \in \mathrm{T}_q\mathrm{M}$ with $|X| < \varepsilon$ and define $Y := \frac{1}{\varepsilon_1}X$. Then $|Y| < \varepsilon_2$, i.e. $Y \in W \subset \tilde{W}$. Thus there exists a geodesic $\tilde{\gamma}\colon [-\varepsilon_1, \varepsilon_1] \to \mathrm{M}$ with $\tilde{\gamma}'(0) = Y$. Now define $\gamma\colon [0,1] \to \mathrm{M}$ by $\gamma(s) = \tilde{\gamma}(\varepsilon_1 s)$. Then γ is a geodesic with $\gamma'(0) = \varepsilon_1\tilde{\gamma}'(0) = \varepsilon_1 Y = X$. □

Definition 45 (Exponential map)**.** $\Omega := \{X \in \mathrm{TM} \mid \exists\, \gamma\colon [0,1] \to \mathrm{M}$ *geodesic with* $\gamma'(0) = X\}$. *Define* $\exp\colon \Omega \to \mathrm{M}$ *by* $\exp(X) = \gamma(1)$, *where* $\gamma\colon [0,1] \to \mathrm{M}$ *is the geodesic with* $\gamma'(0) = X$.

Lemma 5. *If $\gamma\colon [0,1] \to \mathrm{M}$ is a geodesic with $\gamma'(0) = X$ then $\gamma(t) = \exp(tX)$ for all $t \in [0,1]$.*

Proof. For $t \in [0,1]$ define $\gamma_t\colon [0,1] \to \mathrm{M}$ by $\gamma_t(s) = \gamma(ts)$. Then $\gamma_t'(0) = tX$, $\gamma_t(1) = \gamma(t)$, γ_t is a geodesic. So $\exp(tX) = \gamma_t(1) = \gamma(t)$. □

Exercise 43. *Show that two isometries $F_1, F_2\colon \mathrm{M} \to \mathrm{M}$ which agree at a point p and induce the same linear mapping from T_pM agree on a neighborhood of p.*

Theorem 49. *Let $p \in \mathrm{M}$. Then there is $\varepsilon > 0$ and an open neighborhood $U \subset \mathrm{M}$ of p such that $B_\varepsilon := \{X \in \mathrm{T}_p\mathrm{M} \mid |X| < \varepsilon\} \subset \Omega$ and $\exp|_{B_\varepsilon}\colon B_\varepsilon \to U$ is a diffeomorphism.*

Proof. From the last lemma we get $d_{0_p}\exp(X) = X$. Here we used the canonical identification between $\mathrm{T}_p\mathrm{M}$ and $\mathrm{T}_{0_p}(\mathrm{TM})$ given by $X \mapsto (t \mapsto tX)$. The claim then follows immediately from the inverse function theorem. □

Definition 46 (Geodesic normal coordinates)**.** $(\exp|_{B_\varepsilon})^{-1}\colon U \to B_\varepsilon \subset \mathrm{T}_p\mathrm{M} \cong \mathbb{R}^n$ *viewed as a coordinate chart is called geodesic normal coordinates near p.*

Exercise 44. *Let M be a Riemannian manifold of dimension n. Show that for each point $p \in \mathrm{M}$ there is a local coordinate $\varphi = (x_1, \dots, x_n)$ at p such that*

$$g\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)\Big|_p = \delta_{ij}, \qquad \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\Bigg|_p = 0.$$

Theorem 50 (Gauss lemma)**.** $\exp|_{B_\varepsilon}$ *maps radii $t \mapsto tX$ in B_ε to geodesics in M. Moreover, these geodesics intersect the hypersurfaces $S_r := \{\exp(X) \mid X \in B_\varepsilon, |X| = r\}$ orthogonally.*

Proof. This follows by the last lemma and the rope construction of spheres. □

Definition 47 (Distance)**.** *Let M be a connected Riemannian manifold. Then for $p, q \in \mathrm{M}$ define the distance $d(p,q)$ by*

$$d(p,q) = \inf\{L(\gamma) \mid \gamma\colon [0,1] \to \mathrm{M} \text{ smooth with } \gamma(0) = p,\ \gamma(1) = q\}.$$

Exercise 45. *a) Is there a Riemannian manifold* (M, g) *which has finite diameter (i.e. there is an* m *such that all points* $p, q \in \mathrm{M}$ *have distance* $d(p, q) < m$*) and there is a geodesic of infinite length without self-intersections?*

b) Find an example for a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ *but which has no geodesic of infinite length.*

Definition 48 (Metric space)**.** *A metric space is a pair* (X, d) *where* X *is a set and* $d\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$ *a map such that*

a) $d(p, q) \geq 0$*,* $d(p, q) = 0 \Leftrightarrow p = q$*,*
b) $d(p, q) = d(q, p)$*,*
c) $d(p, q) + d(q, r) \geq d(p, r)$*.*

Is a Riemannian manifold (with its distance) a metric space. Symmetry is easy to see: If $\gamma\colon [0,1] \to \mathrm{M}$ is a curve from p to q, then $\tilde{\gamma}(t) := \gamma(1-t)$ is a curve from q to p and $L(\tilde{\gamma}) = L(\gamma)$. For the triangle inequality we need to concatenate curves. So let $\gamma\colon [0, 1 \to \mathrm{M}]$ be a curve from p to q and $\tilde{\gamma}\colon [0,1] \to \mathrm{M}$ be a curve from q to r. Though the naive concatenation is not smooth we can stop for a moment and then continue running: Let $\varphi\colon [0,1] \to [0,1]$ be smooth monotone function such that $\varphi(0) = 0$, $\varphi(1) = 1$ and φ' vanishes on $[0, \varepsilon) \cup (1-\varepsilon, 1]$ for some $\varepsilon > 0$ sufficiently small. Then define for γ from p to q and $\tilde{\gamma}$ from q to r

$$\hat{\gamma}(t) = \begin{cases} \gamma(\varphi(2t)), & \textit{for } t \in [0, 1/2), \\ \tilde{\gamma}(\varphi(2t-1)), & \textit{for } t \in [1/2, 1]. \end{cases}$$

Then $L(\hat{\gamma}) = L(\gamma) + L(\tilde{\gamma})$. For every $\varepsilon > 0$ we find γ and $\tilde{\gamma}$ such that

$$L(\gamma) \leq d(p, q) + \varepsilon, \quad L(\tilde{\gamma}) \leq d(q, r) + \varepsilon.$$

Thus by concatenation we obtain a curve $\hat{\gamma}$ from q to r such that $L(\hat{\gamma}) \leq d(p, q) + d(q, r) + 2\varepsilon$. Thus $d(p, r) \leq d(p, q) + d(q, r)$. Certainly, $L(\gamma) \geq 0 \rightsquigarrow d(p, q) \geq 0$ and $d(p, p) = 0$. So the only part still missing is that $p = q$ whenever $d(p, q) = 0$.

Theorem 51. *Let* $p \in \mathrm{M}$ *and* $f\colon B_\varepsilon \to U \subset \mathrm{M}$ *be geodesic normal coordinates at* p*. Then*

$$d(p, \exp(X)) = |X|, \quad \textit{for } |X| \leq \varepsilon.$$

Moreover, for $q \notin U$*,* $d(p, q) > \varepsilon$*.*

Proof. Choose $0 < R < \varepsilon$. Take $\gamma\colon [0,1] \to \mathrm{M}$ with $\gamma(0) = p$, $\gamma(t) := \exp(tX)$ with $|X| = R$. Let $q := \gamma(1) = \exp(X)$. Then $L(\gamma) = R$. In particular, $d(p, q) \leq R$. Now, choose $0 < r < R$ and let $\gamma\colon [0,1] \to \mathrm{M}$ be any curve with $\gamma(0) = p$ and $\gamma(1) = q$. Define a to be the smallest $t \in [0,1]$ such that there is Y such that $\gamma(t) = \exp(Y)$, $|Y| = r$. Define b to be the smallest $t \in [0, 1]$, $a < b$, such that there is Z such that $\gamma(b) = \exp(Z)$, $|Z| = R$. Now find $\xi\colon [a, b] \to \mathrm{TM}$ such that $r < |\xi(t)| < R$ for all $t \in (a, b)$, $|\xi(a)| = r$, $|\xi(b)| = R$ and $\exp(\xi(t)) = \gamma(t)$ for all $t \in [a, b]$. Define $\rho\colon [a, b] \to \mathrm{M}$ by $\rho := |\xi|$ and $\nu\colon [a, b] \to \mathrm{M}$ by $\xi =: \rho\nu$. Claim: $L(\gamma|_{[a,b]}) \geq R - r$. Afterwards: $L(\gamma) \geq R - r$ for all such $r > 0$. Hence $L(\gamma) \geq R$ and thus $d(p, q) = R$. Let us prove the claim: For all $t \in [a, b]$ we have

$$\gamma'(t) = d\exp(\xi'(t)) = d\exp(\rho'(t)\nu(t) + \rho(t)\nu'(t)) = \rho'(t)d\exp(\nu(t)) + \rho(t)d\exp(\nu'(t)).$$

By the Gauss lemma we get then

$$|\gamma'(0)|^2 = |\rho'(t)d\exp(\nu(t))|^2 + |\rho(t)d\exp(\nu'(t))|^2 \geq |\rho'(t)|^2 \underbrace{|d\exp(\nu(t))|^2}_{=1} = \rho'(t)^2.$$

Thus we have
$$L(\gamma|_{[a,b]}) = \int_a^b |\rho'| \geq \int_a^b \rho' = \rho|_a^b = R - r.$$
Certainly, we can have equality only for $\nu' = 0$. This yields the second part. □

Corollary 4. *A Riemannian manifold together with its distance function is a metric space.*

Corollary 5. *Let $\gamma\colon [0, L] \to \mathrm{M}$ be an arclength-parametrized geodesic. Then there is $\varepsilon > 0$ such that $d(\gamma(0), \gamma(t)) = t$ for all $t \in [0, \varepsilon]$.*

The first variational formula says: If $\gamma\colon [a, b] \to \mathrm{M}$ is a smooth length-minimizing curve, i.e. $L(\gamma) = d(\gamma(a), \gamma(b))$, then γ is a geodesic. To see this, choose a function $\rho\colon [a, b] \to \mathbb{R}$ with $\rho(s) > 0$ for all $s \in (a, b)$ but $\rho(a) = 0 = \rho(b)$. Then there is $\varepsilon > 0$ such that $\alpha\colon (-\varepsilon, \varepsilon) \times (a, b) \to \mathrm{M}$, $\alpha(t, s) = \exp(t\rho(s)\gamma''(s))$. Without loss of generality we can assume that $|\gamma'| = 1$. Then
$$0 = \frac{d}{dt}\Big|_{t=0} L(\gamma) = \underbrace{\langle \gamma', \rho\gamma''\rangle|_a^b}_{=0} - \int_a^b \langle \rho\gamma'', \gamma''\rangle = -\int_a^b \rho|\gamma''|^2$$
for all such ρ. Thus we conclude $\gamma'' = 0$ and so γ is a geodesic. We need a slightly stronger result. For preparation we give the following exercise:

Exercise 46. $d(p, q) = \inf\{L(\gamma) \mid \gamma\colon [a, b] \to \mathrm{M}$ *piecesewise smooth*, $\gamma(a) = p$, $\gamma(b) = q\}$.

Theorem 52. *Let $\gamma\colon [0, L] \to \mathrm{M}$ be a continuous piecewise-smooth curve with $|\gamma'| = 1$ (whenever defined) such that $L(\gamma) = d(\gamma(0), \gamma(L))$. Then γ is a smooth geodesic.*

Proof. Let $0 = s_0 < \cdots < s_k = L$ be such that $\gamma|_{[s_{i-1}, s_i]}$ is smooth, $i = 1, \ldots, k$. The above discussion then shows that the parts $\gamma|_{[s_{i-1}, s_i]}$ are smooth geodesics. We need to show that there are no kinks. Let $j \in \{1, \ldots, k-1\}$ and $X := \gamma'|_{[s_{j-1}, s_j]}(s_j)$, $\tilde{X} = \gamma'|_{[s_j, s_{j+1}]}(s_j)$. Claim: $X = \tilde{X}$. Define $Y = \tilde{X} - X$ and choose any variation γ_t of γ which does nothing on $[0, s_{j-1}] \cup [s_{j+1}, L]$. Then
$$0 = \frac{d}{dt}\Big|_{t=0} L(\gamma_t) = \sum_j \frac{d}{dt}\Big|_{t=0} L(\gamma_t|_{[s_{j-1}, s_j]}) = \langle Y, X\rangle - \langle Y, \tilde{X}\rangle = |\tilde{X} - X|^2.$$
Thus $\tilde{X} - X = 0$. □

15. Complete Riemannian manifolds

Definition 49 (Complete Riemannian manifold)**.** *A Riemannian manifold is called complete if* exp *is defined on all of* TM*, or equivalently: every geodesic can be extended to* $\mathbb{R}$*.*

Theorem 53 (Hopf and Rinow)**.** *Let* M *be a complete Riemannian manifold, $p, q \in \mathrm{M}$ Then there is a geodesic $\gamma\colon [0, L]$ with $\gamma(0) = p$, $\gamma(L) = q$ and $L(\gamma) = d(p, q)$.*

Proof. Let $\varepsilon > 0$ be such that $\exp|_{B_\varepsilon}$ is a diffeomorphism onto its image. Without loss of generality, assume that $\delta < d(p, q)$. Let $0 < \delta < \varepsilon$ and set $S := \exp(S_\delta)$, where $S_\delta = \partial B_\delta$. Then $f\colon S \to \mathbb{R}$ given by $f(r) = d(r, q)$ is continuous. Since S is compact, there is $r_0 \in S$ where f has a minimum, i.e.
$$d(r_0, q) \leq d(r, p) \text{ for all } r \in S.$$
Then $r_0 = \gamma(\delta)$, where $\gamma\colon \mathbb{R} \to \mathrm{M}$ with $\gamma(0) = p$. Define
$$d(S, q) := \inf\{d(r, q) \mid r \in S\}.$$

Then $d(S,q) = d(r_0,q)$. Every curve $\eta\colon [a,b] \to \mathrm{M}$ from p to q has to hit S: There is $t_0 \in [a,b]$ with $\eta(t_0) \in S$. Moreover,

$$L(\eta) = L(\eta|_{[a,t_0]}) + L(\eta|_{[t_0,b]}) \geq \delta + d(S,q) = \delta + d(r_0,q).$$

So $d(p,q) \geq \delta + d(r,q)$. On the other hand, the triangle inequality yields $d(p,q) \leq d(p,r_0) + d(r_0,q) = \delta + d(r_0,q)$. Thus $d(\gamma(\delta),q) = d(p,q) - \delta$.

Define statement $A(t)$: "$d(\gamma(t),q) = d(p,q) - t$." So we know $A(\delta)$ is true. We want to show that also $A(d(p,q))$ is true. Define

$$t_0 := \sup\{t \in [0, d(p,q)] \mid A(t) \ \textit{true}\}.$$

Assume that $t_0 < d(p,q)$. Claim: $A(t_0)$ is true. This is because there is a sequence $t_1, t_2, \ldots,$ with $\lim_{n\to\infty} t_n = t_0$ and $A(t_n)$ true, i.e. $f(t_n) = 0$ where $f(t) = d(\gamma(t),q) - (d(p,q) - t)$. Clearly, f is continuous. Thus $f(t_0) = 0$, too.

Now let $\tilde{\gamma}$ be a geodesic constructed as before but emanating from $\gamma(t_0)$. With the same argument as before we then get again

$$d(\tilde{\gamma}(\tilde{\delta}),q) = d(\tilde{\gamma}(0),q) - \tilde{\delta}.$$

Now, since $A(t_0)$ is true, we have

$$d(p,q) \leq d(p,\tilde{\gamma}(\tilde{\delta})) + d(\tilde{\gamma}(\tilde{\delta}),q) = d(p,\tilde{\gamma}(\tilde{\delta})) + d(\tilde{\gamma}(0),q) - \tilde{\delta} = d(p,\tilde{\gamma}(\tilde{\delta})) + d(p,q) - t_0 - \tilde{\delta}.$$

There obviously is a piecewise-smooth curve from p to $\tilde{\gamma}(\tilde{\delta})$ of length $t_0+\tilde{\delta}$. So $d(p,\tilde{\gamma}(\tilde{\delta})) \leq t_0+\tilde{\delta}$. Hence $d(p,\tilde{\gamma}(\tilde{\delta})) = t_0 + \tilde{\delta}$. Hence this piecewise-smooth curve is length minimizing and in particular it is smooth, i.e. there is no kink and thus we have $\tilde{\gamma}(\tilde{\delta}) = \gamma(t_0 + \tilde{\delta})$.

Now we have $d(\gamma(t_0+\tilde{\delta}),q) = d(\gamma(t_0),q) - \tilde{\delta} = d(p,q) - (t_0+\tilde{\delta})$. Thus $A(t_0+\tilde{\delta})$ is true, which contradicts the definition of t_0. So $A(d(p,q))$ is true. □

Theorem 54. *For a Riemannian manifold* M *the following statements are equivalent:*

a) M *is complete Riemannian manifold.*
b) All bounded closed subsets of M *are compact.*
c) (M, d) *is a complete metric space.*

Proof. *a)* ⇒ *b)*: Let $A \subset \mathrm{M}$ be closed and bounded, i.e. there is $p \in \mathrm{M}$ and $c \in \mathbb{R}$ such that $d(p,q) \leq c$ for a all $p,q \in A$. Look at the ball $B_c \subset \mathrm{T}_p\mathrm{M}$. Hopf-Rinow implies then that $A \subset \exp(B_c)$. Hence A is a closed subset of a compact set and thus compact itself. *b)* ⇒ *c)* is a well-known fact: Any Cauchy sequence $\{p_n\}_{n\in\mathbb{N}}$ is bounded and thus lies in bounded closed set which then is compact. Hence $\{p_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which then converges to the limit of $\{p_n\}_{n\in\mathbb{N}}$. *c)* ⇒ *a)*: Let $\gamma\colon [0,\ell] \to \mathrm{M}$ be a geodesic.

$$T := \sup\{t \geq \ell \mid \gamma \textit{ can be extended to } [0,T]\}.$$

We want to show that $T = \infty$. Define $p_n := \gamma(T - \frac{1}{n})$. Then $\{p_n\}_{n\in\mathbb{N}}$ defines a Cauchy sequence which thus has a limit point $p := \lim_{n\to\infty} p_n$. Thus γ extends to $[0,T]$ by setting $\gamma(T) := p$. Thus γ extends beyond T. which contradicts the definition of T. □

Exercise 47. *A curve γ in a Riemannian manifold* M *is called divergent, if for every compact set $K \subset \mathrm{M}$ there exists a $t_0 \in [0,a)$ such that $\gamma(t) \notin K$ for all $t > t_0$. Show:* M *is complete if and only if all divergent curves are of infinite length.*

Exercise 48. *Let* M *be a complete Riemannian manifold, which is not compact. Show that there exists a geodesic $\gamma\colon [0,\infty) \to \mathrm{M}$ which for every $s > 0$ is the shortest path between $\gamma(0)$ and $\gamma(s)$.*

Exercise 49. *Let* M *be a compact Riemannian manifold. Show that* M *has finite diameter, and that any two points* $p, q \in \mathrm{M}$ *can be joined by a geodesic of length* $d(p,q)$.

16. Sectional curvature

Definition 50 (Sectional curvature)**.** *Let* M *be a Riemannian manifold,* $p \in \mathrm{M}$, $\mathrm{E} \subset \mathrm{T}_p\mathrm{M}$, $\dim \mathrm{E} = 2$, $\mathrm{E} = \operatorname{span}\{X, Y\}$. *Then*

$$K_{\mathrm{E}} := \frac{\langle R(X,Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}$$

is called the sectional the sectional curvature of E.

Exercise 50. *Check that* K_{E} *is well-defined.*

Theorem 55. *Let* M *be a Riemannian manifold,* $p \in \mathrm{M}$, $X, Y, Z, W \in \mathrm{T}_p\mathrm{M}$. *Then*

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle.$$

Proof. The Jacobi identity yields the following 4 equations:

$$\begin{aligned}
0 &= \langle R(X,Y)Z,W\rangle + \langle R(Y,Z)X,W\rangle + \langle R(Z,X)Y,W\rangle,\\
0 &= \langle R(Y,Z)W,X\rangle + \langle R(Z,W)Y,X\rangle + \langle R(W,Y)Z,X\rangle,\\
0 &= \langle R(W,Z)X,Y\rangle + \langle R(X,W)Z,Y\rangle + \langle R(Z,X)W,Y\rangle,\\
0 &= \langle R(X,W)Y,Z\rangle + \langle R(Y,X)W,Z\rangle + \langle R(W,Y)X,Z\rangle.
\end{aligned}$$

□

The following theorem tells us that the sectional curvature completely determine the curvature tensor R.

Theorem 56. *Let* V *be a Euclidean vector space.* $R\colon V \times V \to V$ *bilinear with all the symmetries of the curvature tensor of a Riemannian manifold. For any* 2*-dimensional subspace* $E \subset V$ *with orthonormal basis* X, Y *define* $K_E = \langle R(X,Y)Y, X\rangle$. *Let* $\tilde{R}$ *be another such tensor with* $\tilde{K}_E = K_E$ *for all* 2*-dimensional subspaces* $E \subset V$. *Then* $\tilde{R} = R$.

Proof. $K_E = \tilde{K}_E$ implies $\langle R(X,Y)Y, X\rangle = \langle \tilde{R}(X,Y)Y, X\rangle$ for *all* $X, Y \in V$. We will show that we can calculate $\langle R(X,Y)Z, W\rangle$ for all $X, Y, Z, W \in V$ provided we know $\langle R(X,Y)Y, X\rangle$ for all $X, Y \in V$. Let $X, Y, Z, W \in V$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(s,t) = \langle R(X+sW, Y+tZ)(Y+tZ), X+sW\rangle - \langle R(X+sZ, Y+tW)(Y+tW), X+sZ\rangle.$$

For fixed X, Y, Z, W this is polynomial in s and t. We are only interested in the st term: It is

$$\begin{aligned}
&\langle R(W,Z)Y,X\rangle + \langle R(W,Y)Z,X\rangle + \langle R(X,Z)Y,W\rangle + \langle R(X,Y)Z,W\rangle\\
&\qquad - \langle R(Z,W)Y,X\rangle - \langle R(Z,Y)W,X\rangle - \langle R(X,W)Y,Z\rangle - \langle R(X,Y)Z,W\rangle\\
&= 4\langle R(X,Y)Z,W\rangle + 2\langle R(W,Y)Z,X\rangle - 2\langle R(Z,Y)W,X\rangle\\
&= 4\langle R(X,Y)Z,W\rangle + 2\langle R(W,Y)Z + R(Y,Z)W,X\rangle\\
&= 4\langle R(X,Y)Z,W\rangle - 2\langle R(Z,W)Y,X\rangle\\
&= 6\langle R(X,Y)Z,W\rangle.
\end{aligned}$$

□

Corollary 6. *Let* M *be a Riemannian manifold and* $p \in$ M*. Suppose that* $K_E = K$ *for all* $E \subset \mathrm{T}_p\mathrm{M}$ *with* $\dim E = 2$*. Then*

$$R(X,Y)Z = K(\langle Z,Y\rangle X - \langle Z,X\rangle Y).$$

Proof. Define $\tilde{R}$ by this formula. Then $\tilde{R}(X,Y)$ is skew in X,Y and

$$\langle \tilde{R}(X,Y)Z, W\rangle = K(\langle Y,Z\rangle\langle X,W\rangle - \langle Z,X\rangle\langle Y,W\rangle)$$

is skew in Z,W. Finally,

$$\begin{aligned}&\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y \\ &\qquad = K(\langle Z,Y\rangle X - \langle Z,X\rangle Y + \langle X,Z\rangle Y - \langle X,Y\rangle Z + \langle Y,X\rangle Z - \langle Y,Z\rangle X) = 0.\end{aligned}$$

and if $X,Y \in \mathrm{T}_p\mathrm{M}$ is an orthonormal basis then

$$\tilde{K}_E = K\langle\langle Y,Y\rangle X - \langle Y,X\rangle Y, X\rangle = K.$$

□

17. Jacobi fields

Let $\gamma\colon [0,L] \to \mathrm{M}$ be a geodesic and $\alpha\colon (-\varepsilon,\varepsilon)\times[0,L] \to \mathrm{M}$ be a geodesic variation of γ, i.e. $\gamma_t = \alpha(t,.)$ is a geodesic for all $t \in (-\varepsilon,\varepsilon)$. Then the corresponding variational vector field $Y \in \Gamma(\gamma^*\mathrm{TM})$ along γ,

$$Y_s = \frac{\partial\alpha}{\partial t}\Big|_{(0,s)},$$

is called a Jacobi field.

Lemma 6. *Let* α *be a variation of a curve,* $\tilde{\nabla} = \alpha^*\nabla$ *and* $\tilde{R} = \alpha^* R$*. Then*

$$\tilde{\nabla}_{\frac{\partial}{\partial s}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial t}\alpha = \tilde{R}(\frac{\partial}{\partial s},\frac{\partial}{\partial t})\tfrac{\partial}{\partial s}\alpha + \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha.$$

Proof. Since ∇ is torsion-free we have $\tilde{\nabla}_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t}\alpha = \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s}\alpha$. The equation then follows from $[\frac{\partial}{\partial s},\frac{\partial}{\partial t}] = 0$. □

Theorem 57. *A vector field* $Y \in \Gamma(\gamma^*\mathrm{TM})$ *is a Jacobi field if and only if it satisfies*

$$Y'' + R(Y,\gamma')\gamma' = 0.$$

Proof. "⇒": With the lemma above evaluated for $(0,s)$ we obtain

$$Y'' = \tilde{R}(\tfrac{\partial}{\partial s},\tfrac{\partial}{\partial t})\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} + \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} = \tilde{R}(\frac{\partial}{\partial s},\frac{\partial}{\partial t})\tfrac{\partial}{\partial s}\alpha\Big|_{(0,s)} = R(\gamma',Y)\gamma'.$$

"⇐": Suppose a vector field Y along γ satisfies $Y'' + R(Y,\gamma')\gamma' = 0$. We want to construct a geodesic variation α such that $\gamma_0 = \gamma$ and with variational vector field Y. The solution of a linear second order ordinary differential equation Y is uniquely prescribed by $Y(0)$ and $Y'(0)$. In particular the Jacobi fields form a $2n$-dimensional vector space. Denote $p := \gamma(0)$. By the first part of the prove it is enough to show that for each $V,W \in \mathrm{T}_p\mathrm{M}$ there exists a geodesic variation α of γ with variational vector field Y which satisfies $Y(0) = V$ and $Y'(0) = W$: The curve $\eta\colon (-\tilde{\varepsilon},\tilde{\varepsilon}) \to \mathrm{M}$, $\eta(t) = \exp(tV)$ is defined for $\tilde{\varepsilon} > 0$ small enough. Define a parallel vector field $\tilde{W}$ along η with $\tilde{W}_0 = W$. Similarly, let $\tilde{U}$ be parallel along η such that $\tilde{U}_0 = \gamma'(0)$. Now define $\alpha\colon [0,L]\times(-\varepsilon,\varepsilon) \to \mathrm{M}$ by $\alpha(s,t) = \exp(s(\tilde{U}_t + t\tilde{W}_t))$, for $\varepsilon > 0$ small enough. Clearly, α is a geodesic variation of γ. From $\tilde U_t, \tilde W_t \in \mathrm{T}_{\eta(t)}\mathrm{M}$ we get $\gamma_t(0) = \eta(t)$ and hence $Y(0) = \eta'(0) = V$. Moreover, $Y'(0) = \dot\alpha'(0,0) = \nabla_{\frac{\partial}{\partial t}|_{(s,t)=(0,0)}} \alpha' = \nabla_{\frac{\partial}{\partial t}|_{t=0}} (\tilde U_t + t\tilde W_t) = \tilde W_0 = W$. □

Exercise 51. *Show that, as claimed in the previous proof, there is $\varepsilon > 0$ such that for $|t| < \varepsilon$ the geodesic $\gamma_t = \alpha(.,t)$ really lives for time L.*

Trivial geodesic variations: $\gamma_t(s) = \gamma(a(t)s + b(t))$ with functions a and b such that $a(0) = 1$, $b(0) = 0$. Then the variational vector field is just $Y_s = (a'(0)s + b'(0))\gamma'(s)$. Thus $Y' = a'(0)\gamma'$ and hence $Y'' = 0$. Certainly also $R(Y, \gamma') = 0$. Thus Y is a Jacobi field.

Interesting Jacobi fields are orthogonal to γ': Let Y be a Jacobi-field. Then $f\colon [0, L] \to \mathbb{R}$, $f = \langle Y, \gamma'\rangle$. Then $f' = \langle Y', \gamma'\rangle$ and $f'' = \langle Y'', \gamma'\rangle = -\langle R(Y, \gamma')\gamma', \gamma'\rangle = 0$. Thus there are $a, b \in \mathbb{R}$ such that $f(s) = as + b$. In particular, with $V := Y(0)$ and $W := Y'(0)$ we have $f(0) = \langle V, \gamma'\rangle$, $f'(0) = \langle W, \gamma'(0)\rangle$. Then we will have $f \equiv 0$ provided that $V, W \perp \gamma'(0)$. So $\langle Y, \gamma'\rangle \equiv 0$ in this case. This defines a $(2n-2)$-dimensional space of (interesting) Jacobi fields.

Example 5: *Consider $\mathrm{M} = \mathbb{R}^n$. Then Y Jacobi field along $s \mapsto p + sv$ if and only if $Y'' \equiv 0$, i.e. $Y(s) = V + sW$ for parallel vector fields V, W along γ (constant).*

Example 6: *Consider the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and let $p, V, W \in \mathbb{R}^{n+1}$ be orthonormal. Define γ_t as follows*

$$\gamma_t(s) := \cos s\, p + \sin s\,(\cos t\, V + \sin t\, W).$$

Then $Y_s = \sin s\, W$ is a Jacobi field and thus

$$-\sin s\, W = Y''(s) = -R(Y(s), \gamma'(0))\gamma'(0) = -\sin s R(W, \gamma')\gamma'.$$

Thus $W = R(W, \gamma')\gamma'$. Evaluation for $s = 0$ then yields $W = R(W, V)V$. In particular, if $E = \operatorname{span}\{V, W\} \subset \mathrm{T}_p\mathbb{S}^n$, then $K_E = \langle R(W, V)V, W\rangle = 1$.

18. Second variational formula

Theorem 58 (Second variational formula)**.** *Let $\alpha\colon (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times [0, L] \to \mathrm{M}$ be a 2-parameter variation of a geodesic $\gamma\colon [0, L] \to \mathrm{M}$, i.e. $\alpha(0, 0, s) = \gamma(s)$, with fixed endpoints, i.e. $\alpha(u, v, 0) = \gamma(0)$ and $\alpha(u, v, L) = \gamma(L)$ for all $u, v \in (-\varepsilon, \varepsilon)$. Let*

$$X_s = \frac{\partial\alpha}{\partial u}\Big|_{(0,0,s)}, \quad Y_s = \frac{\partial\alpha}{\partial v}\Big|_{(0,0,s)}, \quad \gamma_{u,v}(s) := \alpha(u, v, s).$$

Then

$$\frac{\partial^2}{\partial u \partial v} E(\gamma_{u,v})(0,0) = -\int_0^L \langle X, Y'' + R(Y, \gamma')\gamma'\rangle.$$

Remark 14: *Actually, that is an astonishing formula. Since the left hand side is symmetric in u and v, the right hand side must be symmetric in X and Y. Let's check this first directly: Let $X, Y \in \Gamma(\gamma^*\mathrm{TM})$ such that $X_0 = 0 = Y_0$ and $X_L = 0 = Y_L$. Then with partial integration we get*

$$\int_0^L \langle X, Y'' + R(Y, \gamma')\gamma'\rangle = \int_0^L \langle X, Y''\rangle + \int_0^L \langle X, R(Y, \gamma')\gamma'\rangle = -\int_0^L \langle X', Y'\rangle + \int_0^L \langle X, R(Y, \gamma')\gamma'\rangle,$$

which is symmetric in X and Y.

Proof. First,

$$\frac{\partial}{\partial u}E(\gamma_{u.v}) = \tfrac{1}{2}\frac{\partial}{\partial u}\int_0^L \langle \tfrac{\partial}{\partial s}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle = \int_0^L \langle \nabla_{\frac{\partial}{\partial u}}\tfrac{\partial}{\partial s}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle = \int_0^L \langle \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial u}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle$$
$$= \langle \tfrac{\partial}{\partial u}\alpha, \tfrac{\partial}{\partial s}\alpha\rangle\Big|_0^L - \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle = -\int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle.$$

Now, let us take the second derivative:

$$\frac{\partial^2}{\partial u\partial v}E(\gamma_{u.v}) = -\frac{\partial}{\partial v}\int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle = -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}}\tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle - \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial v}}\nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle$$
$$= -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}}\tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\tfrac{\partial}{\partial s}\alpha\rangle - \int_0^L \langle \tfrac{\partial}{\partial u}\alpha, \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial v}}\tfrac{\partial}{\partial s}\alpha + R(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial s})\tfrac{\partial}{\partial s}\alpha\rangle.$$

Evaluation at $(u, v) = (0, 0)$ yields

$$\frac{\partial^2}{\partial u\partial v}E(\gamma_{u,v})(0,0) = -\int_0^L \langle \nabla_{\frac{\partial}{\partial v}}\tfrac{\partial}{\partial u}\alpha\Big|_{(u,v)=(0,0)}, \gamma''\rangle - \int_0^L \langle X, Y'' + R(Y, \gamma')\gamma'\rangle.$$

With $\gamma'' = 0$ we obtain the desired result. □

19. Bonnet-Myers's theorem

Missing: Integration on manifolds. Let M be compact Riemannian manifold, $f \in \mathscr{C}^\infty(\mathrm{M})$. Then on can define

$$\int_{\mathrm{M}} f \in \mathbb{R}.$$

If M is any orientable manifold of dimension n and $\omega \in \Omega_0^n(\mathrm{M})$ then one can define

$$\int_{\mathrm{M}} \omega.$$

There is an interesting relation between Topology and geometry (curvature).

Definition 51 (Simply connected)**.** *A manifold* M *is called simply connected if for every smooth map* $\gamma\colon \mathbb{S}^1 \to \mathrm{M}$*,* $\mathbb{S}^1 = \partial D^2$*, there is a smooth map* $f\colon D^2 \to \mathrm{M}$ *such that* $\gamma = f|_{\mathbb{S}^1}$*.*

Theorem 59. *Let* M *be a simply connected complete Riemannian manifold with constant sectional curvature* $K > 0$*. Then* M *is isometric to a round sphere of radius* $r = 1/\sqrt{K}$*.*

Without completeness: only a part of the sphere, **without 'simply connected':** $\mathbb{R}\mathrm{P}^n$ has also constant sectional curvature. Similar with lense spaces: Identify points on $\mathbb{S}^3 \subset \mathbb{C}^2$ that differ by $e^{2\pi i/n}$, $\mathrm{M} = \mathbb{S}^3/\sim$.

Theorem 60. M *simply connected complete, for all sectional curvature* K_E *we have* $\frac{1}{4} < K_E \leq 1$*. Then* M *is homeomorphic to* $\mathbb{S}^n$*.*

Remark 15: *For* $\mathrm{M} = \mathbb{C}\mathrm{P}^n$ *one has* $\frac{1}{4} \leq K_E \leq 1$*.*

Theorem 61 (Gauss-Bonnet)**.** *Let* M *be compact of dimension* 2*. Then there is an integer* $\chi(M) \leq 2$ *such that*

$$\int_{\mathrm{M}} K = 2\pi\chi(M).$$

If M*,* $\tilde{\mathrm{M}}$ *are orientable, then:* $\chi(\mathrm{M}) = \chi(\tilde{\mathrm{M}}) \Leftrightarrow \mathrm{M}, \tilde{\mathrm{M}}$ *diffeomorphic.*

Definition 52 (Scalar curvature)**.** M *Riemannian manifold,* $p \in$ M*,* $G_2(\mathrm{T}_p\mathrm{M})$ *Grassmanian of 2-planes* $E \subset \mathrm{T}_p\mathrm{M}$ *(*$\rightsquigarrow \dim G_2(\mathrm{T}_p\mathrm{M}) = n(n-1)/2$*). Then*

$$\tilde{S} := \tfrac{1}{\mathrm{vol}(G_2(\mathrm{T}_p\mathrm{M}))} \int_{G_2(\mathrm{T}_p\mathrm{M})} K_E$$

is called the scalar curvature.

Definition 53 (Ricci curvature)**.** M *Riemannian,* $p \in$ M*,* $X \in \mathrm{T}_p\mathrm{M}$*,* $|X| = 1$*,* $\mathbb{S}^{n-2} \subset X^\perp \subset \mathrm{T}_p\mathrm{M}$*. Then*

$$\widetilde{\mathrm{Ric}}(X,X) = \tfrac{1}{\mathrm{vol}(\mathbb{S}^{n-2})} \int_{\mathbb{S}^{n-2}} K_{\mathrm{span}\{X,Y\}} dY$$

is called Ricci curvature.

Let us try something simpler: Choose an orthonormal basis $Z_1, \dots, Z_n$ of $\mathrm{T}_p\mathrm{M}$ with $Z_1 = X$ and define

$$\mathrm{Ric}(X,X) := \tfrac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, X)X, Z_i\rangle = \tfrac{1}{n-1} \sum_{i=2}^{n} K_{\mathrm{span}\{X,Z_i\}}.$$

Then with $AZ := R(Z,X)X$ defines an endomorphism of $\mathrm{T}_p\mathrm{M}$ and

$$\mathrm{Ric}(X,X) := \tfrac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, X)X, Z_i\rangle = \tfrac{1}{n-1} \sum_{i=1}^{n} \langle AZ_i, Z_i\rangle = \tfrac{1}{n-1}\mathrm{tr}(A).$$

Thus $\mathrm{Ric}(X,X)$ does not depend on the choice of the basis.

Definition 54.

$$\mathrm{Ric}(X,Y) = \tfrac{1}{n-1}\mathrm{tr}(Z \mapsto R(Z,X)Y).$$

Theorem 62. $\mathrm{Ric}_p \colon \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \to \mathbb{R}$ *is symmetric.*

Proof.

$$\mathrm{Ric}(X,Y) = \tfrac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, X)Y, Z_i\rangle = \tfrac{1}{n-1} \sum_{i=1}^{n} \langle R(Z_i, Y)X, Z_i\rangle = \mathrm{Ric}(Y,X).$$

□

Now we have two symmetric bilinear forms on each tangent space, $\langle .,. \rangle$ and Ric.

Theorem 63 (*without proof*)**.** $\widetilde{\mathrm{Ric}}(X,X) = \mathrm{Ric}(X,X)$.

Definition 55. *Define* $\mathrm{ric}_p \colon \mathrm{T}_p\mathrm{M} \to \mathrm{T}_p\mathrm{M}$ *by* $\langle \mathrm{ric}_p X, Y\rangle := \mathrm{Ric}(X,Y)$.

$\rightsquigarrow$ Eigenvalues $\kappa_1, \dots, \kappa_n$ of ric_p (and eigenvectors) provide useful information.

Definition 56. $Z_1, \dots, Z_n$ *orthonormal basis of* $\mathrm{T}_p\mathrm{M}$*. Then define*

$$S(p) := \tfrac{2}{n(n-1)} \sum_{i<j} \langle R(Z_i, Z_j)Z_j, Z_i\rangle.$$

$$\begin{aligned}\tfrac{1}{n}\sum_{j=1}^{n} \mathrm{Ric}(Z_j, Z_j) &= \tfrac{1}{n}\sum_{j=1}^{n} \tfrac{1}{n-1} \sum_{i\neq j} \langle R(Z_i, Z_j)Z_j, Z_i\rangle = \tfrac{1}{n(n-1)} \sum_{j=1}^{n}\sum_{i\neq j} \langle R(Z_i, Z_j)Z_j, Z_i\rangle \\ &= \tfrac{2}{n(n-1)} \sum_{i<j} \langle R(Z_i, Z_j)Z_j, Z_i\rangle = \tfrac{1}{n}\sum_{j=1}^{n} \langle \mathrm{ric}_p Z_j, Z_j\rangle = \tfrac{1}{n}\mathrm{tr}(\mathrm{ric}_p).\end{aligned}$$

Theorem 64 (*without proof*)**.** $\tilde{S}(p) = S(p)$.

Definition 57 (Diameter)**.** *Let* M *be a Riemannian manifold. Then*

$$\operatorname{diam}(\mathrm{M}) := \sup\{d(p,q) \mid p,q \in \mathrm{M}\} \in \mathbb{R} \cup \{\infty\}$$

is called the diameter of M*.*

Theorem 65. M *complete* $\rightsquigarrow$ $\operatorname{diam}(\mathrm{M}) < \infty \Leftrightarrow M$ *compact.*

Proof. "$\Rightarrow$": $\operatorname{diam}(\mathrm{M}) < \infty$, then M closed and bounded, thus compact. "$\Leftarrow$": $d\colon \mathrm{M} \times \mathrm{M} \to \mathbb{R}$ is continuous, thus takes its maximum. $\rightsquigarrow \operatorname{diam}(\mathrm{M}) < \infty$. □

Theorem 66 (Bonnet-Myers)**.** M *complete Riemannian manifold,* $\operatorname{Ric}(X,X) \geq \frac{1}{r^2}\langle X, X\rangle$ *all* $X \in \mathrm{TM}$*. Then* $\operatorname{diam}(\mathrm{M}) \leq \pi r$*.*

Proof. Choose $p, q \in \mathrm{M}$. $L := d(p,q) > 0$. By Hopf-Rinow there is an arclength-parametrized geodesic $\gamma\colon [0,L] \to \mathrm{M}$ with $\gamma(0) = p$ and $\gamma(L) = q$. Now choose a parallel orthonormal frame field $X_1, \dots, X_n$ along γ with $X_1 = \gamma'$. Define $Y_i \in \Gamma(\gamma^*\mathrm{TM})$ by $Y_i(s) = \sin(\frac{\pi s}{L})X_i(s)$. Define variations $\tilde{\alpha}_i\colon (-\varepsilon,\varepsilon) \times [0,L] \to \mathrm{M}$ of γ by $\alpha_i(t,s) = \exp(tY_i(s))$, $\gamma_t^i = \tilde{\alpha}_i(t,.)$. With $\alpha_i(u,v,s) := \tilde{\alpha}_i(u+v,s)$ we have

$$\frac{\partial \alpha_i}{\partial u}\bigg|_{(0,0,s)} = Y_i(s) = \frac{\partial \alpha_i}{\partial v}\bigg|_{(0,0,s)}.$$

Then we use the second variational formula of length: If $g\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$, $g(t) = L(\gamma_t)$, then g has a global minimum at $t = 0$, i.e. $0 \leq g''(0)$. Thus

$$0 \leq g''(0) = \tfrac{\partial^2}{\partial u \partial v} L(\gamma^i_{u,v}) = -\int_0^L \langle Y_k, Y_k'' + R(Y_k,\gamma')\gamma'\rangle.$$

Since $Y_k(s) = \sin(\frac{\pi s}{L})X_k$, we have $Y_k''(s) = -(\frac{\pi}{L})^2 \sin(\frac{\pi s}{L})X_k(s)$. Thus, for each k,

$$(\tfrac{\pi}{L})^2 \int_0^L \sin^2(\tfrac{\pi}{L}) = -\int_0^L \langle Y_k'', Y_k\rangle \geq \int_0^L \langle R(Y_k,\gamma')\gamma', Y_k\rangle = \int_0^L \sin^2(\tfrac{\pi s}{L})\langle R(X_k,\gamma')\gamma', X_k\rangle.$$

By assumption $\operatorname{Ric}(X,X) \geq \frac{1}{r^2}\langle X, X\rangle$. Thus summing over $k = 2, \dots, n$ we get

$$\begin{aligned}\tfrac{n-1}{r^2} \int_0^L \sin^2(\tfrac{\pi s}{L}) &\leq (n-1)\int_0^L \sin^2(\tfrac{\pi s}{L})\operatorname{Ric}(\gamma',\gamma') \\ &= \sum_{k=2}^n \int_0^L \sin^2(\tfrac{\pi s}{L})\langle R(X_k,\gamma')\gamma', X_k\rangle \\ &\leq (n-1)(\tfrac{\pi}{L})^2 \int_0^L \sin^2(\tfrac{\pi}{L}).\end{aligned}$$

Then, since $\int_0^L \sin^2(\frac{\pi s}{L}) > 0$, we get $L \leq \pi r$. □