TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I - WS 2016/2017

http://www3.math.tu-berlin.de/geometrie/Lehre/WS16/GeometryI/

Exercise Sheet 3

Exercise 1: Ceva's Theorem. (2 pts)

Consider a triangle $\Delta(A_1, A_2, A_3)$ in the plane an let P_{12}, P_{23}, P_{31} be some points on the lines (A_1A_2), (A_2A_3), (A_3A_1), respectively, different from the vertices A_i. Set $O = (A_1P_{23}) \cap (A_2P_{31})$ and $R_{12} = (A_1A_2) \cap (A_3O)$. Show:

$$\frac{l(A_1P_{12})}{l(P_{12}A_2)} \cdot \frac{l(A_2P_{23})}{l(P_{23}A_3)} \cdot \frac{l(A_3P_{31})}{l(P_{31}A_1)} = \mathrm{cr}(A_1, P_{12}, A_2, R_{12})\,.$$

Exercise 2: Cross-ratios. (6 pts)

(i) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines through a point P in $\mathbb{R}\mathrm{P}^2$. Let g be another line *not* containing P. Then the cross-ratio $\mathrm{cr}(\ell_1, \ell_2, \ell_3, \ell_4)$ of the four lines is defined as the cross-ratio of the four intersection points $\ell_i \cap g$. Show that the cross-ratio of the lines does not depend on the choice of g.

(ii) Let E_1, E_2, E_3, E_4 be four planes in $\mathbb{R}\mathrm{P}^3$ which all pass through one line ℓ. Show that the following two definitions of the cross-ratio $\mathrm{cr}(E_1, E_2, E_3, E_4)$ are equivalent. In particular, show that they do not depend on the choice of ℓ' and E', respectively.

(a) Let ℓ' be a line that is not contained in any of the four planes and let P_i be the intersection of ℓ' with E_i. Let $\mathrm{cr}(E_1, E_2, E_3, E_4) \coloneqq \mathrm{cr}(P_1, P_2, P_3, P_4)$.

(b) Let E' be a plane that does not contain the line ℓ. Let ℓ_i be the line of intersection of E_i and E'. Let $\mathrm{cr}(E_1, E_2, E_3, E_4) \coloneqq \mathrm{cr}(\ell_1, \ell_2, \ell_3, \ell_4)$.

Exercise 3: Complex cross-ratio. (4 pts)

Consider a decomposition of the complex projective line $\mathbb{C}\mathrm{P}^1$ into $\mathbb{C}$ and ∞. Show that the cross-ratio of the four corners of a square in $\mathbb{C}$ in cyclic order is equal to -1.

Exercise 4: Projective involutions. (4 pts)

(i) Let $f : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be a projective involution with two fixed points $P, Q \in \mathbb{R}\mathrm{P}^1$. Show that for all $X \in \mathbb{R}\mathrm{P}^1$ the pair $\{X, f(X)\}$ harmonically separates $\{P, Q\}$, i.e. $\mathrm{cr}(X, P, f(X), Q) = -1$.

(ii) Show that for any two points $P, Q \in \mathbb{R}\mathrm{P}^1$ there is a unique projective involution of $\mathbb{R}\mathrm{P}^1$ with fixed points P and Q.

Due Thursday, November 17, before the lecture.