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Geometry I - WS 2016/2017

http://www3.math.tu-berlin.de/geometrie/Lehre/WS16/GeometryI/

Exercise Sheet 10

Exercise 1: Hyperbolic space as inside of a quadric. (4 pts)

Consider the Lorentz vector space $\mathbb{R}^{n,1}$ ($n \geq 2$), i.e. the real vector space $\mathbb{R}^{n+1}$ with scalar product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1}$. The corresponding quadric $\mathcal{Q}$ is given by the zero-set of the quadratic form $q(x) := \langle x, x\rangle$. In the lecture we defined the projective model of hyperbolic space $\mathbf{H}^n_{pr} = \{[x] \in \mathbb{RP}^n \mid q(x) < 0\}$. A point $P \in \mathbb{RP}^n$ is *inside* the quadric, if every line through P intersects the quadric in exactly two different points. (The points of the quadric itself are not inside, since the tangents intersect the quadric in only one point.)
Show: $P = [x] \in \mathbb{RP}^n$ is inside $\mathcal{Q}$ if and only if $q(x) < 0$. In other words the projective model of hyperbolic space corresponds to the inside of the quadric.

Exercise 2: Lorentz product. (3 pts)

Let $p, q \in \mathbb{H}^n$. Show that $\langle p, q\rangle \leq -1$ with equality if and only if $p = q$.

Exercise 3: Metric in the hyperbolic plane. (4 pts)

Let $l, l_1, l_2 \subset \mathbb{H}^2$ be three lines in the hyperbolic plane with normals n, n_1, n_2 satisfying $|\langle n_1, n_2\rangle| > 1$. Further let $x \in \mathbb{H}^2$ be a point that does not lie on l and $\langle n, n\rangle = \langle n_1, n_1\rangle = \langle n_2, n_2\rangle = 1$ and $\langle x, x\rangle = \langle x_1, x_1\rangle = \langle x_2, x_2\rangle = -1$. Show:

(i) The distance $d(x, l)$ of x to l satisfies $|\langle n, x\rangle| = \sinh d(x, l)$.

(ii) The distance $d(l_1, l_2)$ between the lines l_1 and l_2 satisfies $|\langle n_1, n_2\rangle| = \cosh d(l_1, l_2)$. It is equal to $d(x_1, x_2)$ where $x_1 = l_1 \cap l_3$ and $x_2 = l_2 \cap l_3$ and l_3 is the unique hyperbolic line orthogonal to l_1 and l_2.

Exercise 4: Klein disk model. (3 pts)

Let $O = (0, 0)$ and $P = (x, 0)$ with $x > 0$ be points in the Klein disk model of the hyperbolic plane, and assume $d(O, P) = t$. Show that $x = \tanh(t)$.

Exercise 5: Halfplanes in the hyperbolic plane. (2 pts)

How many disjoint hyperbolic halfplanes are there in a hyperbolic plane?

Due Thursday, January 19, before the lecture.