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Geometry I - WS 2016/2017

http://www3.math.tu-berlin.de/geometrie/Lehre/WS16/GeometryI/

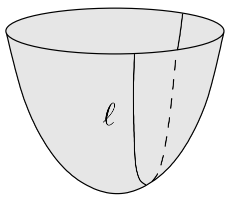
Exercise Sheet 13

Exercise 1: Hyperbolic reflections and inversions. (4 pts)

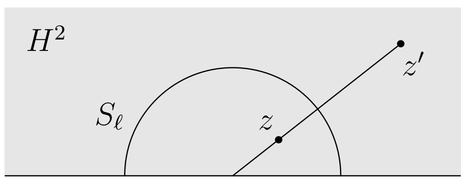
Consider a hyperbolic line ℓ in the hyperboloid model given by

$$\ell = \{x \in \mathrm{P}(\mathbb{R}^{2,1}) \,|\, \langle x, n \rangle = 0, \langle x, x \rangle = -1, \text{ and } x_3 > 0\},$$

where $n \in \mathrm{P}(\mathbb{R}^{2,1})$ with $\langle n, n \rangle = 1$ is the unit normal of the line.
In the Poincaré half-plane model H^2, the hyperbolic line ℓ is represented by an arc of the circle S_ℓ orthogonal to the boundary of the halfplane.
Show that the (hyperbolic) reflection in ℓ is given by the inversion in the circle S_ℓ.



$\langle n, n \rangle = 1, \quad x \mapsto x' = x - 2\langle n, x \rangle n$



$z \mapsto z' = c + \frac{r^2}{|z-c|^2}(z - c)$

Exercise 2: Equi-distant sets in halfplane model. (4 pts)

(i) Show that the transformations $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by

$$z \mapsto \frac{az + b}{cz + d}, \qquad \text{with } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0$$

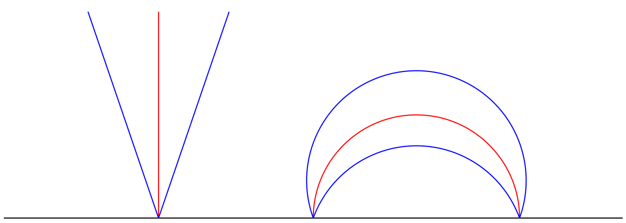
map circles and lines to circles and lines.

(ii) For a line l in the hyperbolic plane and $\delta > 0$, denote

$$E = \left\{ p \in H^2 \mid d(p, l) = \delta \right\}$$

the set of equi-distant points to l.

Show that in the halfplane model E consists of two lines or two circular arcs at constant angle to the hyperbolic line as shown in the figure below.



Exercise 3: Hyperbolic motions in the Poincaré disc. (4 pts)

Show that all orientation preserving hyperbolic motions of the Poincaré disc are of the form

$$f(z) = e^{i\varphi}\frac{z - z_0}{1 - z\bar{z}_0},$$

with $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and $\varphi \in [0, 2\pi)$.

Hint: You already know the isometries in the halfplane model.

Exercise 4: Normalization of two circles. (4 pts)

Show that by inversion in a circle a pair of circles can be mapped either to a pair of straight lines, or to a pair of concentric circles.

Due Thursday, February 09, before the lecture.