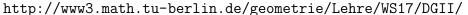
## TECHNISCHE UNIVERSITÄT BERLIN Institut für Mathematik

Pinkall / Knöppel





WS 17

## Differential Geometry II: Analysis and Geometry on Manifolds

## Exercise Sheet 1

(Manifolds)

due 30.10.2017

Exercise 1 3 points

Let X be a topological space,  $x \in X$  and  $n \ge 0$ . Show that the following statements are equivalent:

- i) There is a neighborhood of x which is homeomorphic to  $\mathbb{R}^n$ .
- ii) There is a neighborhood of x which is homeomorphic to an open subset of  $\mathbb{R}^n$ .

Exercise 2 6 points

Let  $n \in \mathbb{N}$  and  $X := \mathbb{R}^{n+1} \setminus \{0\}$ . The quotient space  $\mathbb{R}P^n = X/_{\sim}$  with equivalence relation given by

$$x \sim y : \iff x = \lambda y, \quad \lambda \in \mathbb{R}$$

is called the *n*-dimensional real projective space. Let  $\pi: X \to \mathbb{R}P^n$  denote the canonical projection  $x \mapsto [x]$ . For  $i = 0, \dots, n$ , we define  $U_i := \pi(\{x \in X \mid x_i \neq 0\})$  and  $\varphi_i: U_i \to \mathbb{R}^n$  by

$$[x_0,\ldots,x_n]\mapsto (x_0/x_i,\ldots,\widehat{x_i},\ldots,x_n/x_i).$$

Show that

- a)  $\pi$  is an open map, i.e. maps open sets in X to open sets in  $\mathbb{R}P^n$ ,
- b) the maps  $\varphi_i$  are well-defined and  $\{(U_i, \varphi_i)\}_{i \in I}$  is a smooth atlas of  $\mathbb{R}P^n$ ,
- c)  $\mathbb{R}P^n$  is compact. Hint: Note that the restriction of  $\pi$  to  $\mathbb{S}^n$  is surjective.

Exercise 3 6 points

Let M and N be topological manifolds of dimension m and n, respectively. Show that their Cartesian product  $M \times N$  is a topological manifold of dimension m + n. Show further that, if  $\{(U_{\alpha}, \varphi_{\alpha})\}_{\alpha \in A}$  is a smooth atlas of M and  $\{(V_{\beta}, \psi_{\beta})\}_{\beta \in B}$  is a smooth atlas of N, then  $\{(U_{\alpha} \times V_{\beta}, \varphi_{\alpha} \times \psi_{\beta})\}_{(\alpha,\beta) \in A \times B}$  is a smooth atlas of  $M \times N$ . Here  $\varphi_{\alpha} \times \psi_{\beta} \colon U_{\alpha} \times V_{\beta} \to \varphi_{\alpha}(U_{\alpha}) \times \psi_{\beta}(V_{\beta})$  is defined by  $\varphi_{\alpha} \times \psi_{\beta}(p,q) := (\varphi_{\alpha}(p), \psi_{\beta}(q))$ .