Technische Universität Berlin
Institut für Mathematik
Pinkall / Knöppel

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 3

(Diffeomorphisms, immersions, submersions)

due 13.11.2017

Exercise 1 **5 points**

Let $\mathrm{Sym}(3) \cong \mathbb{R}^6$ denote the set of real symmetric 3×3 matrices. Let

$$M := \{P \in \mathrm{Sym}(3) \mid P^2 = P, \, tr(P) = 1\}.$$

Show that $M \subset \mathrm{Sym}(3)$ is a submanifold diffeomorphic to $\mathbb{R}\mathrm{P}^2$.

Exercise 2 **5 points**

Let $f\colon N \to M$ be a smooth immersion. Prove: If f is moreover a *topological embedding*, i.e. its restriction $f\colon N \to f(N)$ is a homeomorphism between N and $f(N)$ (with its subspace topology), then $f(N)$ is a smooth submanifold of M.

Exercise 3 **5 points**

Let $X := \mathbb{C}^2 \setminus \{0\}$. The complex projective plane is the quotient space $\mathbb{C}\mathrm{P}^1 = X/_\sim$, where the equivalence relation is given by

$$\psi \sim \tilde{\psi} :\Leftrightarrow \lambda\psi = \tilde{\psi}, \quad \lambda \in \mathbb{C}.$$

Consider $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, then the *Hopf fibration* is the map

$$\pi\colon \mathbb{S}^3 \to \mathbb{C}\mathrm{P}^1, \quad \psi \mapsto [\psi].$$

Show: For each $p \in \mathbb{C}\mathrm{P}^1$ the *fiber* $\pi^{-1}(\{p\})$ is a submanifold diffeomorphic to $\mathbb{S}^1$.