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Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 9

(Curvature, isometries, geodesics)

due 08.01.2017

Definition: Let $f\colon M \to N$ be a diffeomorphism and $X \in \Gamma(\mathrm{T}M)$. Then the *push forward* $f_*X \in \Gamma(\mathrm{T}N)$ *of* X is defined by $f_*X := df \circ X \circ f^{-1}$.

Exercise 1 **5 points**

Let (M, g) and (N, h) be Riemannian manifolds with Levi-Civita connections ∇ and $\tilde{\nabla}$, respectively. Let $f\colon M \to N$ be an *isometry*, i.e. a diffeomorphism such that $f^*h(X, Y) := h(df(X), df(Y)) = g(X, Y)$ for all $X, Y \in \Gamma(TM)$. Show that

$$\tilde{\nabla}_{f_*X} f_*Y = f_* \left(\nabla_X Y\right).$$

Show further that γ is a geodesic in M if and only if $\tilde{\gamma} = f \circ \gamma$ is a geodesic in N.

Exercise 2 **5 points**

Let $\langle .,. \rangle$ be the Euclidean metric on $\mathbb{R}^n$ and $B := \{x \in \mathbb{R}^n \mid |x|^2 < 1\}$. For $k \in \{-1, 0, 1\}$ define

$$g_k|_x := \frac{4}{(1 + k|x|^2)^2} \langle .,. \rangle.$$

Show that for the curvature tensors R_k of the Riemannian manifolds (B, g_{-1}), $(\mathbb{R}^n, g_0)$ and $(\mathbb{R}^n, g_1)$ and for every $X, Y \in \mathbb{R}^n$ the following equation holds:

$$g_k(R_k(X, Y)Y, X) = k\big(g_k(X, X)g_k(Y, Y) - g_k(X, Y)^2\big).$$

Exercise 3 **5 points**

Let M be a Riemannian manifold, $\gamma\colon I \to M$ be a curve which is parametrized with constant speed, and $f\colon M \to M$ be an isometry which fixes γ, i.e. $f \circ \gamma = \gamma$. Furthermore, let

$$\ker(\mathrm{id} - d_{\gamma(t)} f) = \mathbb{R}\dot{\gamma}(t), \ \textit{for all } t.$$

Then γ is a geodesic.