TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. John Sullivan
Jan Techter

Geometry I - WS 2017/2018

http://www3.math.tu-berlin.de/geometrie/Lehre/WS17/GeometryI/

Exercise Sheet 10

Exercise 1: Stereographic projection in $\mathbb{R}^3$. (4 pts)

Consider the stereographic projection of the sphere $S^2 \subset \mathbb{R}^3$ from the north pole e_3 to the plane $z = 0$. Give a brief description and provide a sketch for each of the following questions. Feel free to use colors!

(i) What are the images of the lines of latitude (the circles centered at $\pm e_3$) and of the lines of longitude (the great circles through $\pm e_3$) under stereographic projection?

(ii) Now rotate the pattern on the sphere to consider the circles centered at $\pm e_1$ along with the great circles through $\pm e_1$. What are their images under stereographic projection?

(iii) Consider the horizontal lines in the plane (lines of the form $y = const.$). What are their preimages under stereographic projection?

Exercise 2: Lorentz group $O(1, 1)$. (4 pts)

Consider the orthogonal group $O(1, 1)$ acting on the Lorentz space $\mathbb{R}^{1,1}$ with its scalar product $\langle x, y\rangle = x_1y_1 - x_2y_2$.

(i) Show that any $v \in \mathbb{R}^{1,1}$ with $\langle v, v\rangle = 1$ can be written as $(\pm\cosh t, \sinh t)$ for some $t \in \mathbb{R}$, while any $v \in \mathbb{R}^{1,1}$ with $\langle v, v\rangle = -1$ can be written as $v = (\sinh t, \pm\cosh t)$.

(ii) Consider the family of matrices

$$R_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in O(1, 1).$$

Show that for any $s, t \in \mathbb{R}$ we have $R_sR_t = R_{s+t}$.

(iii) Show that every matrix in $O(1, 1)$ can be written as DR_t where D is one of the four possible diagonal matrices with diagonal entries ± 1 and R_t is as above for some $t \in \mathbb{R}$.

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Exercise 3: . (4 pts)

Denote by $\mathcal{L} = \{v \in \mathbb{R}^{2,1} \mid \langle v, v \rangle = 0\}$ the light cone in $\mathbb{R}^{2,1}$. Let $n \in \mathbb{R}^{2,1}$ be a unit space-like vector (that is, $\langle v, v \rangle = 1$) and let $U = \{v \in \mathbb{R}^{2,1} \mid \langle v, n \rangle = 0\}$ be the plane Lorentz-orthogonal to n.

(i) Show that $\mathcal{L} \cap U$ consists of two lines l_1 and l_2.

(ii) Show that the span of n and l_1 (resp. n and l_2) is a plane in $\mathbb{R}^{2,1}$ tangent to $\mathcal{L}$.

Exercise 4: Orthogonal lines. (4 pts)

Let l_1 and l_2 be hyperbolic lines with normals n_1 and n_2 satisfying $\langle n_1, n_1 \rangle = \langle n_2, n_2 \rangle = 1$. Show that there exists a unique hyperbolic line l_3 such that $l_1 \perp l_3$ and $l_2 \perp l_3$ if and only if $|\langle n_1, n_2 \rangle| > 1$.

Due Wednesday, January 17.