TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Mathematical Physics I, Dr. Matteo Petrera, René Zander WS 19/20
http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 3 (05.11.19)

Due date: **12.11.19**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (6 pts)

Consider the following two one-parameter maps acting on $\mathbb{R}^2$ ($t \in \mathbb{R}$):

$$\Phi_t : (t, (x_1, x_2)) \mapsto \frac{\left(2\,x_1, 2\,x_2 \cos t + \left(1 - x_1^2 - x_2^2\right) \sin t\right)}{1 + x_1^2 + x_2^2 + \left(1 - x_1^2 - x_2^2\right) \cos t - 2\,x_2 \sin t},$$

and

$$\Psi_t : (t, (x_1, x_2)) \mapsto \left(x_1 + t, \frac{x_1 x_2}{x_1 + t}\right).$$

1. Prove that both Φ_t and Ψ_t define a one-parameter Lie group of smooth diffeomorphisms.
2. Compute the infinitesimal generators of Φ_t and Ψ_t and write down their associated IVPs.
3. Construct Ψ_t in terms of its Lie series:

$$(\Psi_t(x_1, x_2))_i = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{v}^k[x_i], \qquad i = 1, 2,$$

   where $\mathbf{v}$ is the infinitesimal generator of Ψ_t computed in 2.
4. Check whether the flows Φ_t and Ψ_t commute.

Exercise 2 (4 pts)

1. Consider the following IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \\ (x_1(0), x_2(0)) \in \mathbb{R}^2, \end{cases}$$

   where $f_1, f_2 \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Note that the ODE

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{1}$$

   can be formally obtained from the original system by dividing the second ODE by the first and then canceling the time differential $\mathrm{d}t$. Let $F(x_1, x_2) = c$, $c \in \mathbb{R}$, be the solution of (1) in implicit form. Prove that F is an integral of motion of the IVP.

Turn over

2. Use the procedure described in 1. to construct an integral of motion in the case

$$f_1(x_1, x_2) := 2\,x_1 - x_1 x_2, \qquad f_2(x_1, x_2) := -9\,x_2 + 3\,x_1 x_2.$$

Sketch the phase portrait in the first quadrant.

Exercise 3 (6 pts)

Fix $n \geq 3$. Let $D \subset \mathbb{R}^n$ be a compact set. On D consider the dynamical system defined through the flow of an n-dimensional IVP whose ODEs are

$$\dot{x}_i = \prod_{j \neq i}^{n} x_j, \qquad i = 1, \ldots, n. \tag{2}$$

1. Prove that the flow is volume preserving.
2. Prove that the functions

$$F_{ij}(x_i, x_j) := x_i^2 - x_j^2, \qquad i, j = 1, \ldots, n.$$

are integrals of motion. How many of these are functionally independent?

3. Define n new coordinates by the transformation $\Phi : x \mapsto y$ defined by

$$y_i := \frac{1}{x_i} \prod_{j \neq i}^{n} x_j, \qquad i = 1, \ldots, n.$$

Prove that the system of ODEs (2) is transformed into the quadratic system

$$\dot{y}_i = y_i \left(-2\,y_i + \sum_{j=1}^{n} y_j \right), \qquad i = 1, \ldots, n. \tag{3}$$

4. Find the functions $K_{ij}(y) := F_{ij}(\Phi^{-1}(y))$. Are the functions K_{ij}, $i, j = 1, \ldots, n.$, integrals of motion of the system of ODEs (3)?

Exercise 4 (4 pts)

Let us denote a sequence $x_k : \mathbb{Z} \to \mathbb{R}$ by x. The iterate of $x \equiv x_k$ is $\widetilde{x} \equiv x_{k+1}$. Consider the dynamical system defined by the planar system of difference equations

$$\begin{cases} \widetilde{x} - x = \alpha(\widetilde{x}\,y + x\,\widetilde{y}), \\ \widetilde{y} - y = -2\,x\,\widetilde{x}, \end{cases} \tag{4}$$

with $\alpha > 0$. The above system defines an explicit map $(x, y) \mapsto (\widetilde{x}, \widetilde{y})$. Without finding the explicit form of the map, use its implicit form, given by (4), to prove that the relation

$$\frac{x^2 + \alpha\,y^2}{1 + \alpha\,x^2} = \frac{\widetilde{x}^2 + \alpha\,\widetilde{y}^2}{1 + \alpha\,\widetilde{x}^2}$$

holds identically on solutions of (4). In other words, the function

$$F(x, y) := \frac{x^2 + \alpha\,y^2}{1 + \alpha\,x^2}$$

is an integral of motion of the discrete dynamical system.