Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, René Zander — WS 19/20

http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 4 (12.11.19)

Due date: **19.11.19**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (5 pts)

Consider the discrete dynamical system defined in terms of iterations of the map $\Phi : \mathbb{R} \setminus \{1/2\} \to \mathbb{R}$ defined by

$$\Phi : x \mapsto \frac{3\,x-2}{2\,x-1}.$$

1. Construct the n-th iteration of Φ.
2. Show that $\lim_{n\to\infty} \Phi^n(x) = 1$, i.e., $x = 1$ is an attracting fixed point.
3. Is the point $x = 1$ asymptotically stable?

Exercise 2 (4 pts)

Consider the following ODE in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = -x_1^3 - 2x_1x_2^2, \\ \dot{x}_2 = x_1^2x_2 - x_2^3. \end{cases}$$

Consider the fixed point $(0,0)$.

1. Is the function
$$F(x_1,x_2) := x_1^2 + x_1^2x_2^2$$
a Lyapunov function? What does it tell you about the stability of $(0,0)$?
2. Is the function
$$G(x_1,x_2) := x_1^2 + x_1^2x_2^2 + x_2^4$$
a Lyapunov function? What does it tell you about the stability of $(0,0)$?

Turn over

Exercise 3 (3 pts)

Consider the following IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_1 - \alpha \sin x_2, \\ \dot{x}_2 = \alpha + x_1^4, \\ (x_1(0), x_2(0)) \in \mathbb{R}^2, \end{cases}$$

with $\alpha > 0$. Prove that the system cannot possess a periodic orbit.

Exercise 4 (4 pts)

Consider a Hamiltonian system of ODEs

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial x_k}, \qquad k = 1, \dots, n \tag{1}$$

with some smooth function $H(x, p) : \mathbb{R}^{2n} \to \mathbb{R}$.

1. Show that the phase space volume is preserved.
2. With the help of the Poincaré recurrence theorem, prove that (1) does not have asymptotically stable fixed points.

Exercise 5 (4 pts)

Consider the following gradient system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = \operatorname{grad}_x G(x), \\ x(0) \in \mathbb{R}^2, \end{cases}$$

where

$$G(x_1, x_2) = x_1^4 + 2x_1^2 x_2^2 + x_2^4 - 4x_2^2 - 2.$$

1. Find the fixed points and study their stability.
2. Sketch the phase portrait.