

Exercise Sheet 5 (19.11.19)

Due date: 26.11.19

- To get the Übungsschein (necessary condition for the oral exam) you need to collect 60% of the total sum of points in each half of the semester. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of 2 or 3 students.
- Please justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1

(4 pts)

Consider the following linear IVP in \mathbb{R}^3 :

$$\begin{cases} \dot{x}_1 = -2x_1 - x_2, \\ \dot{x}_2 = x_1 - 2x_2, \\ \dot{x}_3 = 3x_3, \end{cases}$$

with $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3$.

1. Find the stable, unstable and center subspaces. What can you say about the stability of $(0, 0, 0)$?
2. Sketch the phase portrait.

Exercise 2

(6 pts)

Consider the following linear IVPs in \mathbb{R}^2 :

$$\begin{cases} \dot{x}_1 = -x_1 + x_2, \\ \dot{x}_2 = -x_2, \end{cases} \quad \begin{cases} \dot{x}_1 = 2x_1 + x_2, \\ \dot{x}_2 = 6x_1 + 3x_2, \end{cases}$$

with $(x_1(0), x_2(0)) \in \mathbb{R}^2$. For each of them:

1. Study the stability of the fixed points.
2. Sketch the phase portrait.
3. Find the solution.

TURN OVER!

Exercise 3(5 pts)

Consider the following ODE in \mathbb{R}^3 :

$$\begin{cases} \dot{x}_1 = 3x_2(x_3 - 1), \\ \dot{x}_2 = -x_1(x_3 - 1), \\ \dot{x}_3 = -x_3^3(x_1^2 + 1). \end{cases}$$

1. Linearize the system around the fixed point $(0,0,0)$ and determine the linear stability of the fixed point $(0,0,0)$. What does the Poincaré-Lyapunov theorem say about the stability of $(0,0,0)$ in the nonlinear system?
2. Prove that the fixed point $(0,0,0)$ is stable by finding a (quadratic) Lyapunov function.
3. Prove that the fixed point $(0,0,0)$ cannot be asymptotically stable by restricting the dynamics on the invariant plane $x_3 = 0$.

Exercise 4(5 pts)

Consider the following IVP in \mathbb{R}^2 :

$$\begin{cases} \dot{x} = Ax + \epsilon x \|x\|^2, \\ x(0) \in \mathbb{R}^2 \end{cases}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon \in \mathbb{R}.$$

Study the stability of the fixed point $(0,0)$ if

1. $\epsilon = 0$,
2. $\epsilon < 0$,
3. $\epsilon > 0$.