TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, René Zander WS 19/20

http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 8 (10.12.19)

Due date: **17.12.19**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (6 pts)

Consider two point masses $m_1, m_2 > 0$ in $\mathbb{R}^3$ with time dependent coordinates

$$q^{(j)} := \left(q_1^{(j)}, q_2^{(j)}, q_3^{(j)}\right) \in \mathbb{R}^3, \quad j = 1, 2.$$

Define

$$Q := \left(q^{(1)}, q^{(2)}\right).$$

The total coordinate space is $M = \mathbb{R}^6$. Newton equations of the system form a system of 6 ODEs of the form

$$\ddot{q}^{(1)} = G m_2 \frac{q^{(2)} - q^{(1)}}{\|q^{(2)} - q^{(1)}\|^3}, \quad \ddot{q}^{(2)} = G m_1 \frac{q^{(1)} - q^{(2)}}{\|q^{(1)} - q^{(2)}\|^3},$$

where G is the gravitational constant. Set $G = 1$.

1. Define the vectors

   $$q := q^{(1)} - q^{(2)}, \quad q_{CM} := \frac{m_1 q^{(1)} + m_2 q^{(2)}}{m_1 + m_2}.$$

   Prove that q and q_{CM} obey the ODEs

   $$\frac{1}{m_1 + m_2} \ddot{q} = \operatorname{grad}_q \left(\frac{1}{\|q\|}\right), \quad \ddot{q}_{CM} = 0.$$

2. Prove that the total linear momentum and the total angular momentum,

   $$P(\dot{Q}) := m_1 \dot{q}^{(1)} + m_2 \dot{q}^{(2)}, \quad \ell(Q, \dot{Q}) := m_1 q^{(1)} \times \dot{q}^{(1)} + m_2 q^{(2)} \times \dot{q}^{(2)},$$

   are integrals of motion.

3. Prove that Newton equations are invariant under the change of coordinates

   $$q^{(1)} \mapsto \tilde{q}^{(1)} := A q^{(1)} + a, \quad q^{(2)} \mapsto \tilde{q}^{(2)} := A q^{(2)} + a,$$

   where $A \in \mathbf{O}(3, \mathbb{R}) = \left\{M \in \mathbf{GL}(3, \mathbb{R}) \colon M^T = M^{-1}\right\}$, $a \in \mathbb{R}^3$.

TURN OVER

Exercise 2 (4 pts)

In $\mathbb{R}^n$ consider the system of n Newtonian equations

$$\sum_{i=1}^{n} A_{ik}(q)\ddot{q}_i = \sum_{i,j=1}^{n} \left(\frac{1}{2}\frac{\partial A_{ij}}{\partial q_k} - \frac{\partial A_{ik}}{\partial q_j} \right) \dot{q}_i \dot{q}_j - \frac{\partial U}{\partial q_k},$$

where $k = 1, \ldots, n$, $A := (A_{ij})_{1 \leq i,j \leq n} \in C^2(\mathbb{R}^n, \mathbf{GL}(n, \mathbb{R}))$ is symmetric and positive definite and $U \in C^2(\mathbb{R}^n, \mathbb{R})$ is the potential energy. Prove that the total energy

$$E(q, \dot{q}) := \frac{1}{2}\langle \dot{q}, A(q)\dot{q} \rangle + U(q),$$

is an integral of motion.

Exercise 3 (3+3+4 pts)

1. Fix $T, \alpha > 0$. Consider the functional $\psi : K \to \mathbb{R}$ defined by

   $$\psi(\gamma) := \int_0^T \dot{q}^2 \, \mathrm{d}t,$$

   where K is the space of all C^1-curves

   $$\gamma = \left\{ (t, q) \, : \, q = q(t), \, q \in C^1([0, T], \mathbb{R}), \, q(0) = 0, \, q(T) = \alpha \right\}.$$

   Find an extremal point of ψ. Is this extremal point a candidate to be a maximum or a minimum?

2. Consider the functional $\psi : K \to \mathbb{R}$ defined by

   $$\psi(\gamma) := \int_0^1 \sqrt{q^2 + \dot{q}^2} \, \mathrm{d}t,$$

   where K is the space of all C^1-curves

   $$\gamma = \left\{ (t, q) \, : \, q = q(t), \, q \in C^1([0, 1], \mathbb{R}), \, q(0) = 0, \, q(1) = 1 \right\}.$$

   Write down the Euler-Lagrange equation.
   Prove that $\psi(\gamma) > 1$ for all $\gamma \in K$.

3. Consider the functional $\psi : K \to \mathbb{R}$ defined by

   $$\psi(\gamma) := \int_1^2 \frac{1}{t} \sqrt{1 + \dot{q}^2} \, \mathrm{d}t,$$

   where K is the space of all C^1-curves

   $$\gamma = \left\{ (t, q) \, : \, q = q(t), \, q \in C^1([1, 2], \mathbb{R}), \, q(1) = 0, \, q(2) = 1 \right\}.$$

   Find an extremal point of ψ.