Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, René Zander WS 19/20

http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 9 (17.12.19)

Due date: **07.01.20**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (4 pts)

Consider a point with mass $m = 1$ moving in $\mathbb{R}^3$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) + \alpha(q_1\dot{q}_2 - q_2\dot{q}_1), \qquad \alpha > 0.$$

1. Write down the Euler-Lagrange equations
2. Show that the system is invariant under rotations about the q_3-axis
3. Use the Noether Theorem to find the integral of motion corresponding to the above symmetry. Verify explicitly that the integral of motion is a conserved quantity.

Exercise 2 (6 pts)

1. Compute the Legendre transformation of the function
$$F(q) := \frac{1}{2}q^2 + \frac{1}{3}q^3, \qquad q \in \mathbb{R}.$$
2. Compute the Legendre transformation of the function
$$F(q) := \frac{1}{2}\langle q, A\,q\rangle + c, \qquad q \in \mathbb{R}^n,$$
where $A \in \mathbf{GL}(n, \mathbb{R})$ is symmetric and positive definite and $c \in \mathbb{R}^n$ is a constant vector.
3. Let $F^*(p)$ be the Legendre transformation of a convex function $F \in C^2(\mathbb{R}, \mathbb{R})$.
   (1) Compute the Legendre transformation of $G(q) := F(q) + \alpha\,q$, $\alpha > 0$.
   (2) Compute the Legendre transformation of $G(q) := \alpha\,F(q) + \beta$, $\alpha, \beta > 0$.

TURN OVER

Exercise 3 (5 pts)

Consider a particle of mass $m > 0$ in $\mathbb{R}^3$ moving under the influence of a central potential energy $U : (0, \infty) \to \mathbb{R}$,

$$U(r) := -\alpha \frac{e^{-kr}}{r}, \qquad \alpha, k > 0,$$

where $r = \|x\|$, for $x \in \mathbb{R}^3$. Prove that for sufficiently small values of the angular momentum there exists a stable closed orbit.
(Hint: Construct the effective potential energy.)

Exercise 4 (5 pts)

Consider a one-dimensional mechanical system describing the motion of a point (mass $m = 1$) under the influence of a potential energy

$$U(q) := \frac{q^4}{\alpha + \beta q^2},$$

where α, β are parameters.

1. Determine the values of α and β for which one can decide by using the Dirichlet theorem that the origin $q = 0$ is a stable fixed point.
2. Linearize the dynamical system around this stable fixed point and determine the frequency of small oscillations.

Christmas Exercise (optional) (5 additional pts)

Consider a heavy flexible cable of length L hanging between two points $(\pm\ell, h)$, $h > 0$, $L > 2\ell$, on a vertical (x, y)-plane (one can think at telephone poles and hanging telephone lines). It hangs in a curve that looks like a parabola, in fact, it is not.
It turns out that the cable has a potential energy which is given by the functional

$$U(y) := \rho g \int_{-\ell}^{\ell} y\sqrt{1 + (y')^2}\mathrm{d}x, \quad y' := \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Here ρ and g are respectively the lineal density of the cable and the constant of gravitational acceleration (fix $\rho g = 1$). To solve the problem one has to minimize the functional U over the set of all functions $y \in C^1(\mathbb{R}, R)$ with $(y(-\ell), y(\ell)) = (h, h)$ under the constraint that the length of the cable is the constant L.

1. Introduce a Lagrange multiplier $\lambda \in \mathbb{R}$ due to the constraint and write down the Euler-Lagrange equation of the problem.
2. Reduce the ODE obtained in (1) to a first-order ODE of the form

   $$(y')^2 = \frac{(y + \lambda)^2}{c} - 1,$$

   where $c \in \mathbb{R}$ is a constant of integration.
3. Integrate the ODE obtained in (2). What is the curve describing the shape of the cable?