Exercise Sheet 11 (14.01.20)

Due date: **21.01.20**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.

- Please work in fixed groups of *2 or 3 students*.

- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.

- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (5 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^4$ consider a discrete dynamical system defined by iterations of the map $G \colon \mathbb{R}^4 \setminus \{q_1 q_2 = 1\} \to \mathbb{R}^4$ given by:

$$
\begin{cases}
\widetilde{q}_1 = -p_2 + \dfrac{2q_1}{1 - q_1 q_2}, \\[2mm]
\widetilde{q}_2 = -p_1 + \dfrac{2q_2}{1 - q_1 q_2}, \\[2mm]
\widetilde{p}_1 = q_2, \\[1mm]
\widetilde{p}_2 = q_1.
\end{cases}
$$

1. Prove that G is a symplectic transformation.

2. Prove that the functions

$$
\begin{aligned}
F_1(q_1, q_2, p_1, p_2) &:= q_1 p_1 - q_2 p_2, \\
F_2(q_1, q_2, p_1, p_2) &:= q_1 q_2 + p_1 p_2 - q_1 q_2 p_1 p_2 - q_1 p_1 - q_2 p_2,
\end{aligned}
$$

   are two functionally independent integrals of motion.

3. Prove that F_1 and F_2 are in involution.

Exercise 2 (5 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^6$ consider a point of unit mass with Hamiltonian

$$
\mathcal{H}(q, p) := \sqrt{\alpha^2 + \langle p, p \rangle} + \langle a, q \rangle, \quad \alpha > 0,
$$

where $a \in \mathbb{R}^3$ is a constant vector.

1. Derive the canonical Hamiltonian equations.

2. Construct the Lagrangian.

3. Derive the Euler-Lagrange equations.

Turn over

Exercise 3 (4 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^6$ consider a point of mass $m = 1$ with Hamiltonian

$$\mathcal{H}(q, p) := \frac{\langle p, p \rangle}{2} + U(\|q\|),$$

where $U = U(\|q\|)$ is a central potential. The *Runge-Lenz vector* is defined by the formula

$$A(q, p) := p \times \ell(q, p) + U(\|q\|) q,$$

where $\ell(q, p) := q \times p$ is the angular momentum.

1. Prove that ℓ is an integral of motion.

2. Prove the following formula:

$$\{A_i(q, p), \mathcal{H}(q, p)\} = p_i \Big(\|q\| U'(\|q\|) + U(\|q\|) \Big).$$

3. Consider the *Kepler potential*

$$U(\|q\|) := -\frac{\alpha}{\|q\|}, \qquad \alpha > 0.$$

Prove that A is an integral of motion.

Exercise 4 (6 pts)

Consider a one-dimensional mechanical system describing the motion of a point (mass $m = 1$) under the influence of a potential energy

$$U(q) = \frac{\alpha}{q} + \log\left(\frac{q^2}{1 + q^2} \right),$$

where $\alpha > 0$ is a parameter.

1. Write down Newton equations and the corresponding dynamical system defined through a system of ODEs in $\mathbb{R}^2$.

2. Find the fixed points and investigate their stability. Draw the graph of the potential energy.

3. Make a qualitative analysis of the motion in the phase space $(q, \dot{q})$. Find the value(s) of α such that there exist periodic orbits.