TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, René Zander WS 19/20

http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 12 (21.02.20)

Due date: **28.02.20**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (10 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^4$ consider the discrete dynamical system defined by iterations of the map $\Phi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$ given by:

$$\widetilde{q}_1 = p_1(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_2, \qquad \widetilde{p}_1 = \frac{q_1}{q_1^2 - q_2^2},$$
$$\widetilde{q}_2 = p_2(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_1, \qquad \widetilde{p}_2 = -\frac{q_2}{q_1^2 - q_2^2}.$$

It can be proved that the map Φ preserves the canonical Poisson brackets.

1. Prove that the functions

$$F_1(q_1, q_2, p_1, p_2) := q_1\, p_1 + q_2\, p_2,$$
$$F_2(q_1, q_2, p_1, p_2) := q_1\, p_2 + q_2\, p_1,$$

   are two functionally independent integrals of motion of Φ.

2. Prove that F_1 and F_2 are in Poisson involution, i.e., $\{\, F_1, F_2 \,\} = 0$

Consider the change of coordinates $\Psi : (q_1, q_2, p_1, p_2) \mapsto (Q_1, Q_2, P_1, P_2)$ defined by:

$$Q_1 = \frac{1}{2}\big(\log(q_1 + q_2) + \log(q_1 - q_2)\big), \qquad P_1 = q_1\, p_1 + q_2\, p_2,$$
$$Q_2 = \frac{1}{2}\big(\log(q_1 + q_2) - \log(q_1 - q_2)\big), \qquad P_2 = q_1\, p_2 + q_2\, p_1.$$

It can be proved that the map Ψ preserves the canonical Poisson brackets.

3. Prove that Ψ allows to *linearize* the map Φ:

$$\widetilde{Q}_1 = Q_1 + \nu_1(F_1, F_2), \qquad \widetilde{P}_1 = P_1,$$
$$\widetilde{Q}_2 = Q_2 + \nu_2(F_1, F_2), \qquad \widetilde{P}_2 = P_2.$$

   Here ν_1 and ν_2 are two functions of the integrals of motion to be determined.

Turn over

Exercise 2 (5 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^2$ consider the parametric transformation

$$(q,p) \mapsto (\widetilde{q},\widetilde{p}) := (-p, q+\alpha p^2), \quad \alpha \in \mathbb{R}.$$

(1) Prove that this is a symplectic transformation by verifying the Lie condition corresponding to a generating function of the first kind, $F_1 = F_1(q,\widetilde{q})$.

(2) Find a generating function of the second kind, $F_2 = F_2(q,\widetilde{p})$, by computing the Legendre transformation of F_1. Verify that F_2 generates the symplectic transformation $(q,p) \mapsto (\widetilde{q},\widetilde{p})$.

Exercise 3 (5 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^2$ consider the parametric transformation

$$(q,p) \mapsto (\widetilde{q},\widetilde{p}) := \left(q\cos\theta - p\frac{\sin\theta}{m\omega}, qm\omega\sin\theta + p\cos\theta\right), \quad \theta \in [0,2\pi).$$

Show that this is a symplectic transformation.

(1) By computing the Poisson bracket $\{\widetilde{q},\widetilde{p}\}$.

(2) By finding a generating function of the transformation equations to express $(p,\widetilde{p})$ in terms of $(q,\widetilde{q})$.