TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Mathematical Physics I, Dr. Matteo Petrera, René Zander — WS 19/20
http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/MP1/

Exercise Sheet 13 (28.01.20)

Due date: **04.02.20**

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points in each half of the semester*. Each exercise sheet has 20 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 or 3 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted*. Please write in a readable way. *Unreadable handwriting will not be corrected*. Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed*.

Exercise 1 (5 pts)

Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and define the exponential of A as

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(1) Prove that, if $[A, B] := AB - BA = 0$, then

$$e^{A+B} = e^A e^B, \quad A, B \in \mathfrak{gl}(n, \mathbb{R}).$$

(2) Prove that

$$\det(e^A) = e^{\mathrm{tr}(A)}.$$

Exercise 2 (5 pts)

In R^3 with coordinates $x := (x_1, x_2, x_3)$ consider the following 2-vector field:

$$X := x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

(1) Prove that X is a Poisson 2-vector field.

(2) Construct the Hamiltonian vector field corresponding to the smooth function

$$H(x) := \frac{1}{2}\left(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\right) + b_1 x_1 + b_2 x_2 + b_3 x_3,$$

with $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$.

Turn over

Exercise 3 (5 pts)

Prove that the implicit midpoint rule

$$(\tilde{q},\tilde{p}) = (q,p) + \epsilon f\left(\frac{\tilde{q}+q}{2},\frac{\tilde{p}+p}{2}\right), \quad \epsilon \in \mathbb{R},$$

is a symplectic integrator, i.e., that the map $(q,p) \mapsto (\tilde{q},\tilde{p})$ is a symplectic transformation, whenever applied to a canonical Hamiltonian system $f = (\nabla_p H, -\nabla_q H)$, with smooth Hamiltonian $H(q,p)\colon \mathbb{R}^{2n} \to \mathbb{R}$.

Exercise 4 (5 pts)

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{X}(M)$ be smooth vector fields and $\omega \in \Omega^2(M)$ a differential 2-form on M. Prove that

$$[\mathbf{v}_1,\mathbf{v}_2] \lrcorner \omega = \mathcal{L}_{\mathbf{v}_1}(\mathbf{v}_2 \lrcorner \omega) - \mathbf{v}_2 \lrcorner \mathcal{L}_{\mathbf{v}_1}\omega.$$

Bonus Exercise (optional) (5 additional pts)

In the canonical Hamiltonian phase space $\mathbb{R}^2$ consider a point of mass $m > 0$ with Hamiltonian

$$\mathcal{H}(q,p) := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \qquad \omega > 0.$$

Introduce the complex variables

$$a_\pm(q,p) := \sqrt{\frac{m\omega}{2}}\left(q \pm \frac{ip}{m\omega}\right), \qquad i^2 = -1.$$

(a) Write the Hamiltonian in terms of $a_\pm$.

(b) Compute the Poisson brackets $\{a_+, a_-\}$ and $\{a_\pm, \mathcal{H}\}$.

(c) Find and solve the equations of motion for the variables $a_\pm$. Determine the general real solution of the equations of motion in terms of the variables q and p.