SELECTED TOPICS IN DISCRETE DIFFERENTIAL GEOMETRY AND VISUALIZATION

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Abstract. Surfaces and curves in space are of central importance in many application areas like Computer Graphics, Physics Simulation or Architecture. While Differential Geometry is concerned with smooth surfaces and curves, within a computer they are always represented as a finite set of points, connected by triangles or line segments. This means that surfaces are really treated as being polyhedral and curves are treated as polygons.

Discrete Differential Geometry is a very active area of research where (instead of looking at discrete objects just as numerical approximations to the smooth ones) the goal is to develop a theory of discrete curves and surfaces that has the same structure as the corresponding smooth theory. Quite often this approach leads to solutions that are "exact" on the discrete level rather than approximations and provide highly efficient new algorithms.

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1. Introduction

An immersion

$$f\colon M \to \mathbb{R}^3$$

is called a *parametrized surface*. Its *Gauss map* is given by

$$N = \tfrac{1}{\sqrt{|f_u|^2+|f_v|^2}} f_u \times f_v.$$

The differential dN of N is tangent to f, since $|N| = 1$ and N is orthogonal to f. Thus, there exists at each $p \in M$ an endomorphism A of $\mathbb{R}^2$, the so called *shape* or *Weingarten operator*, such that

$$dN = df \circ A.$$

The permutability $f_{uv} = f_{vu}$ implies that A is symmetric. Its eigenvalues are called the *principal curvatures* κ_1 and κ_2 of f, its determinant

$$K = \det A = \kappa_1\kappa_2$$

is called the *Gauss curvature*, and half of its trace

$$H = \tfrac{1}{2}\operatorname{tr} A = \tfrac{1}{2}(\kappa_1 + \kappa_2)$$

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is called the *mean curvature* of f at p.

2. Chebyshev nets and the wave equation

2.1. **Definition.** A smooth map $f\colon M \subset \mathbb{R}^2 \to \mathbb{R}^n$ is called a *Chebyshev net* if all parameter lines are arc length parametrized, i.e., $|f_u| = |f_v| = 1$. f is called a *weak Chebyshev net* if and only if there exists a Chebyshev net $\tilde{f}$ such that $f(u,v) = \tilde{f}(\varphi(u), \psi(v))$.

2.2. *Remark.* f is a weak Chebyshev net if and only if $|f_u|_v = |f_v|_u = 0$, i.e., the partial derivative of f with respect to one parameter has constant length along the parameter lines of the other parameter.

2.3. **Planar Chebyshev nets.** In this section we study planar Chebyshev nets. Chebyshev nets in space will be important for the surfaces of constant Gaussian curvature in Section 3.
For a planar (i.e., $\mathbb{R}^2$ valued) Chebyshev net $0 = (|f_u|^2)_v = \langle f_u, f_u\rangle_v = 2\langle f_{uv}, f_u\rangle$ and similarly $0 = \langle f_{uv}, f_v\rangle$ implies that

$$f_{uv} = 0$$

at all points where the partial derivatives f_u and f_v of f are linearly independent.
The partial differential equation $f_{uv} = 0$ is easy to solve.

2.4. **Lemma.** *$f_{uv} = 0$ if and only if there exists functions φ and ψ depending on u and v only such that*

$$f(u,v) = \varphi(u) + \psi(v).$$

This implies in particular that planar immersed Chebyshev nets are of the form $f(u,v) = \varphi(u) + \psi(v)$.

Proof. $(f_u)_v = 0$ implies that there is a function g that depends on u only, such that $f_u(u,v) = g(u)$. Integrating this equation implies for all fixed v_0 that $f(u,v_0) = \psi(v_0) + \int_0^u g$ for some function ψ, thus $f(u,v) = \varphi(u) + \psi(v)$ if φ denotes the primitive of g with $\varphi(0) = 0$. $\square$

2.5. **Wave equation.** In x, t coordinates, where $u = x+t$ and $v = x-t$ one obtains $4f_{uv} = \tilde{f}_{xx} - \tilde{f}_{tt}$, where $\tilde{f}(x,t) = f(u,v)$. Hence, $f_{uv} = 0$ becomes the usual *wave equation*

$$\tilde{f}_{xx} - \tilde{f}_{tt} = 0.$$

In physics courses one often derives the wave equation as the continuum limit of the following discrete problem. Consider a finite number of identical massive balls $f_n \in \mathbb{R}^2$ connected by identical springs. Then Newton's and Hook's law imply that the acceleration $\tilde{f}_{tt}(t,n)$ of the n–th ball satisfies

$$\tilde{f}_{tt}(n,t) = \tfrac{k}{m}(\tilde{f}(n-1,t) - \tilde{f}(n,t)) + (\tilde{f}(n+1,t) - \tilde{f}(n,t)),$$

where m denotes the mass and k the spring constant. In the continuum limit

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{k}{m}\epsilon^2 = c, \quad c\in\mathbb{R}\\
&\tilde{f}_x(x-\tfrac{\epsilon}{2},t) = -\lim_{\epsilon\to 0}\frac{\tilde{f}(n-1,t)-\tilde{f}(n,t)}{\epsilon}, \qquad \text{and}\\
&\tilde{f}_x(x+\tfrac{\epsilon}{2},t) = \lim_{\epsilon\to 0}\frac{\tilde{f}(n+1,t)-\tilde{f}(n,t)}{\epsilon},
\end{aligned}$$

thus $\tilde{f}_{tt} = \tilde{f}_{xx}$.

2.6. **Discrete wave equation.** The discrete wave equation is the piecewise linear case of the continuous wave equation. Consider as in Lemma 2.4

$$f(u,v) = \varphi(u) + \psi(v),$$

with piecewise linear φ and ψ, i.e., linear on intervals $[n, n+1]$, $n \in \mathbb{Z}$. Hence φ and ψ are completely determined by their values on the integers $\mathbb{Z}$ and f solves the *discrete wave equation*

$$f(n,m) + f(n+1,m+1) = f(n+1,m) + f(n,m+1),$$

$n, m \in \mathbb{Z}$. The piecewise linear map f maps fundamental squares of the integer lattice $\mathbb{Z}\times\mathbb{Z}$ onto parallelograms. Since three vertices of a parallelogram determine the fourth vertex, f is determined by its values at time zero and time one, i.e., by $f(n,n)$ and $f(n+1,n)$, $n\in\mathbb{Z}$. These values thus constitute some Cauchy initial data for the discrete wave equation a so called *initial zigzag*.

2.7. **Project.** Implement the discrete wave equation so that the movement of the initial string is visualized. Use linear interpolation to get a smooth movement.

3. K–Surfaces

3.1. **Definition.** The *osculating plane* of a curve $\gamma\colon I\subset\mathbb{R}\to\mathbb{R}^3$ at $s\in I$ is the linear space spanned by $\gamma'(s)$ and $\gamma''(s)$. A curve on a surface is called an *asymptotic line* if the osculating planes of the curve are the tangent planes of the surface, i.e., $\gamma'(s)$ and $\gamma''(s)$ span the tangent plane.

3.2. **Three theorems from Differential Geometry.** We only state the following three theorems from differential geometry.

3.3. **Theorem.** *Every surface with Gauss curvature $K<0$ allows asymptotic line parametrizations $f\colon M\subset\mathbb{R}^2\to\mathbb{R}^3$, i.e., a parametrization whose parameter lines $u\mapsto f(u,v)$ and $v\mapsto f(u,v)$ are asymptotic lines.*

3.4. **Theorem.** *If f is an asymptotic line parametrization of a surface with $K < 0$, then f is weak Chebyshev if and only if its Gauss curvature K is constant.*

Scaling the surface we may restrict to $K = -1$.

3.5. **Corollary.** *If $K = -1$ then the surface allows Chebyshev asymptotic line parametrizations.*

3.6. **Theorem.** *If f is a Chebyshev asymptotic line parametrization of a surface with Gaussian curvature $K = -1$ with Gauss map N, then N is also a Chebyshev net.*

3.7. **Lorentz harmonic maps.** If $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a weak Chebyshev net, i.e., $|f_u|_v = |f_v|_u = 0$.This is equivalent to $f_{uv} = 0$, if $\det(f_u, f_v) \neq 0$. A map satisfying

$$f_{uv} = 0$$

is called *Lorentz harmonic*, since the usual equation for harmonic maps $\tilde{f}_{xx} + \tilde{f}_{tt}$ becomes $\tilde{f}_{xx} - \tilde{f}_{tt} = 0$ if one chooses the standard Lorentzian metric in the coordinates $x = \frac{1}{2}(u+v)$, $t = \frac{1}{2}(u-v)$.
If $f\colon M \subset \mathbb{R}^2 \to \mathbb{R}^3$ is an asymptotic line parametrization with $K = -1$. Then f and its Gauss map N are weak Chebyshev nets. Hence

$$0 = \langle N_u, N_u\rangle_v = 2\langle N_u, N_{uv}\rangle \quad \text{and} \quad 0 = \langle N_v, N_u\rangle_v = 2\langle N_v, N_{uv}\rangle,$$

which is, if N is an immersion, equivalent to

$$N \times N_{uv} = 0.$$

Am S^2–valued map $N\colon M \subset \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ that satisfies this equation will be called *Lorentz harmonic*, since $N \times N_{uv} = 0$ implies that the tangential part of N_{uv} vanishes.
We will now prove that this condition already ensures that N is the Gauss map of a surface of constant Gaussian curvature $K = -1$.

3.8. **Theorem.** *Let $M \subset \mathbb{R}^2$ be simply connected, $N\colon M \to S^2$ with*

$$N \times N_{uv} = 0$$

Then there exists $f\colon M \to \mathbb{R}^3$, uniquely up to translations, such that

$$f_u = N \times N_u, \qquad f_v = -N \times N_v,$$

and f is (away from the points where it fails to be an immersion) an asymptotic line parametrization of a surface of Gaussian curvature $K = -1$ with Gauss map N.

Proof. A map $f\colon M \to \mathbb{R}^3$ with $df = (N \times N_u)du - (N \times N_v)dv$ exists if and only if $(N \times N_u)_v = (N \times N_v)_u = 0$, which is equivalent to $N \times N_{uv} = 0$.
Let $p \in M$ and J_p the endomorphism of $\mathbb{R}^2$ which is mapped by df to the rotation by $\frac{\pi}{2}$, i.e.,

$$df \circ J = N \times df.$$

Then $\det J = 1$ and, since $N \times f_u = -N_u$ and $N \times f_v = N_v$, we get

$$N \times df = dN \circ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = df \circ A \circ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

and hence $K = \det A = -\det J = -1$.
The map f is an asymptotic line parametrization, because $f_{uu} = N \times N_{uu}$ and $f_{vv} = -N \times N_{vv}$ is tangent to f. □

Theorem 3.8 above justifies the following definition.

3.9. **Definition.** A smooth map $f\colon M \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a *K–surface* (shorthand for surface of negative constant Gaussian curvature) if there exists $N\colon M \to S^2$ such that

$$f_u = N \times N_u \quad \text{and} \quad f_v = -N \times N_v.$$

3.10. *Remark.* Note that this definition allows that f and N have singularities, i.e., points at which f or N fails to be an immersion.

4. Discrete Lorentz Harmonic Maps

In 3.7 we saw that Lorentz harmonic maps in S^2 are, away from singular points, the same as weak Chebyshev nets in S^2. That implies that small coordinate quadrilaterals are spherical parallelograms. This property translates easily into the discrete situation, and as we know from Theorem 3.8 it also captures a characterizing property of K–surfaces.

4.1. **Spherical parallelograms.** Let $N_d, N_r, N_u, N_l \in S^2$ be a non degenerated spherical quadrilateral whose edges are shorter than $\frac{\pi}{2}$. Such four points form a spherical parallelogram (a quadrilateral such that opposite edges have the same length) if and only if $(N_u + N_d) \times (N_l + N_r) = 0$, i.e., rotation by π about $(N_u + N_d)$ or $(N_l + N_r)$ maps the quadrilateral onto itself.
The restriction to edges shorter than $\frac{\pi}{2}$ ensures that the entire quadrilateral lies in one hemisphere.

4.2. **Definition.** A discrete map $N\colon \mathbb{Z}^2 \to S^2$ is called *Lorentz harmonic* if and only if

$$(N_u + N_d) \times (N_l + N_r) = 0$$

and all edges are shorter than $\frac{\pi}{2}$.

4.3. **Notation.** For discrete maps subscripts denote points in $\mathbb{Z}^2$. Special subscripts are $d = (m, n)$, $u = (m + 1, n - 1)$, $l = (m, n - 1)$, $r = (m + 1, n)$ for some $m, n \in \mathbb{Z}$. The letters express the fact that if one reflects $\mathbb{Z}^2$ at the line that intersects the first coordinate axes at the angle $\frac{\pi}{4}$ (i.e., displays discrete uv–coordinates in xt–coordinates) then the vertices of the coordinate quadrilaterals are naturally identified by their positions (d)own, (u)p, (l)eft, and (r)ight. Often, e.g,, in Definition 4.2, equations are ment to hold for all admissible $m, n \in \mathbb{Z}$.

4.4. **Definition.** A discrete map $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ such that opposite sides of the quadrilaterals f_d, f_r, f_u, f_l have the same length is called a *weak Chebyshev net.*

With this definitions we may reformulate Statement 8.1 as follows.

4.5. **Lemma.** *If all edges of $N\colon \mathbb{Z}^2 \to S^2$ are shorter than $\frac{\pi}{2}$, then N is discrete Lorentz harmonic if and only if N is weak Chebyshev.*

To solve the equation $(N_u + N_d) \times (N_l + N_r) = 0$ for the fourth point N_u we introduce quaternions.

4.6. **Definition.** The real four dimensional algebra $\mathbb{H} = \{\, r + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid r, x, y, z \in \mathbb{R} \,\}$ with the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ is called the (non–commutative) field of *quaternions.* We identify $\mathbb{R}^3$ with the set of *imaginary quaternions* $\operatorname{Im}\mathbb{H} = \{\, x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid x, y, z \in \mathbb{R} \,\}$. The *conjugate* $\bar{q}$ of a quaternion $q \in \mathbb{H}$ is given by $\overline{r + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}} = r - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ and the length of a quaternion is $|q| = \sqrt{q\bar{q}}$.

4.7. **Exercise.** Check that

a) $|qp| = |q||p|$ for all $q, p \in \mathbb{H}$,
b) $vw = -\langle v, w\rangle + v \times w$, for all $v, w \in \operatorname{Im}\mathbb{H}$, and
c) $q^2 = -1$ is equivalent to $q \in \operatorname{Im}\mathbb{H}$ and $|q| = 1$.

4.8. **Theorem.** *Let $q \in \mathbb{H} \setminus \{0\}$ and let $v \in \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ and $\alpha \in \mathbb{R}$ such that $|v| = 1$ and $q = (\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} v)|q|$. Then*

$$\mathbb{R}^3 \ni y \mapsto qyq^{-1} \in \mathbb{R}^3$$

is a rotation about v by the angle α. The rotation determines q up to multiplication by a real number.

Proof. The map is well defined since $\bar{y} = -y$ and $q^{-1} = q|q|^{-2}$ implies $\overline{qyq^{-1}} = -qyq^{-1}$. The rotation of y about v by the angle α is given by

$$R(y) = \langle y, v\rangle v + \cos(\alpha)(y - \langle y, v\rangle v) + \sin(\alpha) v \times y.$$

On the other hand

$$\begin{aligned} qyq^{-1} &= qy\bar{q}|q|^{-2} \\ &= \left(\cos\left(\tfrac{\alpha}{2}\right) + \sin\left(\tfrac{\alpha}{2}\right) v\right) y \left(\cos\left(\tfrac{\alpha}{2}\right) - \sin\left(\tfrac{\alpha}{2}\right) v\right) \\ &= \left(\cos^2\left(\tfrac{\alpha}{2}\right) y - \sin^2\left(\tfrac{\alpha}{2}\right) vyv\right) + \sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)(vy - yv) \end{aligned}$$

use $vy - yv = 2v \times y$ and $vyv = -2\langle v, y\rangle v - yvv = -2\langle v, y\rangle v + y$

$$\begin{aligned} &= (\cos^2\left(\tfrac{\alpha}{2}\right) - \sin^2\left(\tfrac{\alpha}{2}\right))y + 2\sin^2\left(\tfrac{\alpha}{2}\right)\langle v, y\rangle v + \sin(\alpha) v \times y \\ &= R(y) \end{aligned}$$

□

4.9. **Corollary.** *A discrete Lorentz harmonic map $N\colon \mathbb{Z}^2 \to S^2$ is obtained from an initial zigzag by*

$$N_u = (N_l + N_r)N_d(N_l + N_r)^{-1}.$$

4.10. **Project.** Implement an algorithm to construct a Lorentz harmonic map from an initial zigzag with jReality. Visualize the time evolution $t \in]-\infty, \infty[$ of masses $i = 0, \dots n$ at $(i+t, i-t)$ coupled by rubber band.

5. Discrete K–Surfaces

In analogy to Definition 3.9 we define discrete K–surfaces as follows.

5.1. **Definition.** A map $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ is called a *discrete K–surface* if and only if there exists a discrete map $N\colon \mathbb{Z}^2 \to S^2$ such that $f_r - f_d = N_d \times N_r$ and $f_l - f_d = -N_d \times N_l$. N is called the *Gauss map* of f.

Otherwise said, edges of f are the cross product of the edges of N with either one of the adjacent vertices. The definition of K–surfaces is justified by the following analoguous theorem to Theorem 3.8.

5.2. **Theorem.** *Let $N\colon \mathbb{Z}^2 \to S^2$ be a discrete map and assume that the edges of the quadrilaterals $N_d, N_r, N_u, N_l \in S^2$ are shorter than $\frac{\pi}{2}$. Then there exists a discrete K–surface $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$, i.e.,*

$$f_r - f_d = N_d \times N_r \qquad \text{and} \qquad f_l - f_d = -N_d \times N_l. \tag{5.1}$$

with Gauss map N if and only if N is Lorentz harmonic, i.e.,

$$(N_u + N_d) \times (N_l + N_r) = 0. \tag{5.2}$$

Proof. The formulas 5.1 yield two formulas that involve f_u:

$$f_u - f_l = N_l \times N_u \qquad \text{and} \qquad f_u - f_r = -N_r \times N_u.$$

These yield the same value for f_u if and only if

$$N_l \times N_u + N_r \times N_u = f_r - f_l = N_d \times N_r + N_d \times N_l = 0,$$

which is equivalent to 5.2. □

In the continuous case an asymptotic line parametrization is characterized by the second partial derivative f_{uu} and f_{vv} being tangent to f, cf. 3.1. The discrete version is the following.

5.3. **Definition.** A map $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ is called a *discrete asymptotic line parametrization* if a vertex f_{mn} of f and its four adjacent vertices $f_{(m,n-1)}$, $f_{(m+1,n)}$, $f_{(m,n+1)}$, and $f_{(m-1,n)}$ lie in a plane.

5.4. **Theorem.** *Let $f\colon \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete K–surface. Then f is a weak Chebyshev asymptotic line parametrization.*

Proof. The formulas 5.1 show that the four adjacent edges of a vertex f_{mn} are all orthogonal to the Gauss map at this vertex N_{mn}.
Lemma 4.5 implies that N is a weak Chebyshev net, and $|f_u - f_l| = |N_l \times N_u| = |\sin(\angle(N_l, N_u))|$ implies that f is also a weak Chebyshev net. □

5.5. **Project.** Implement an algorithm to construct a discrete K-surfaces from an initial zigzag.

Discrete K–surfaces are very stable in the sense that they look almost periodic. This behavior is due to an infinite sequence of conservation laws.

6. Special Initial Conditions for Discrete K–Surfaces

The Gauss map N of a discrete K–Surface is, by Corollary 4.9 completely determined by an *initial zigzag*, i.e., the values at time zero $N(n, n)$ and time one $N(n+1, n)$. Suppose that we start with a closed zigzag, i.e., $N(n, n)$ and $N(n + 1, n)$ are periodic in n with the same period. Then the following initial conditions ensure that the corresponding K–surface is also a cylinder, because the symmetry of the initial condition rules out the translational periods that may possibly occur when Theorem 3.8 is applied.

(1) At time $t = 0$ the polygon is collapsed to a point, i.e., $N(n, n)$ is constant. At time $t = 1$ the center of mass of the polygon lies on the axis of the fixed point. This then implies that the discrete K–surface f corresponding to N contains a *planar strip* along the curve $f(n, n)$, i.e., all tangent planes along that curve are equal.
(2) The polygon is initially at rest, i.e., vertices N_d where $d = (n, n)$ at time $t = 0$ are in the center of the great circle through the adjacent vertices N_l and N_r at time $t = 1$, because then N_l, N_d, N_r, N_d form a spherical parallelogram, which implies that $N_u = N_d$, i.e., vertices at time $t = 0$ and $t = 2$ coincide.
Linear dependence of $N_l + N_r$ and N_d implies
$$\begin{aligned} f_r - f_d &= N_d \times N_r = N_d \times (N_r - (N_r + N_l)) \\ &= -N_d \times N_l = f_l - f_d. \end{aligned}$$
Hence $f_r = f_l$, which means that the surface has a *cone point* at time $t = 1$.

6.1. **Project.** Implement an algorithm to construct a discrete K-surfaces from the special initial conditions.

7. Napier's Analogy

In order to better understand the geometry of a discrete Lorentz harmonic map, i.e., a quadmesh consisting of spherical parallelograms we

like to derive a fundamental formula of spherical trigonometry, which is known as Napier's Analogy.

7.1. **Theorem** (Napier's Analogy)**.** *Consider a spherical triangle, i.e., the intersection of three hemispheres whose sides have length a, b, and c with opposite angles α, β, and γ. Then*

$$\frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)} = \tan\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)$$

7.2. *Remark.* Usually there are a lot of Napier's Analogies which can be derived from the one above applying it to the triangles which are obtained taking the complement of one of the defining hemispheres or interchanging the role of angles and side lengths.

7.3. **Discrete moving frames.** We like to give a proof in the spirit of discrete differential geometry.
Recall, that $F = (\gamma, T, N)\colon I \to \mathrm{SO}(3)$ is called an *orthonormal moving frame* of an oriented spherical curve $\gamma\colon I \subset \mathbb{R} \to S^2$ if $T = \frac{\gamma'}{|\gamma'|}$. Identifying the standard basis of $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ with the quaternions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, cf. 4.6, one obtains

$$\gamma = F\mathbf{i}, \qquad T = F\mathbf{j}, \qquad N = F\mathbf{k}.$$

Let $\gamma\colon \mathbb{Z} \to S^2$ be a discrete oriented curve. At each vertex γ_m one has an incoming tangent vector T_{2m} and an outgoing tangent vector T_{2m+1}. Using the representation of rotations in $\mathbb{R}^3$ in Theorem 4.8 we call $F_m \in \mathbb{H}$ a *frame* of γ if

$$\begin{aligned}
\gamma_m &= F_{2m}\mathbf{i}F_{2m}^{-1} = F_{2m+1}\mathbf{i}F_{2m+1}^{-1},\\
T_m &= F_m\mathbf{j}F_m^{-1},\\
N_m &= F_m\mathbf{k}F_m^{-1}.
\end{aligned}$$

The outgoing frame F_{2m+1}, at the vertex γ_m at which the edges form an angle of α, is he incoming frame F_{2m} rotated about γ_m by α, i.e.,

$$F_{2m+1} = F_{2m}\left(1 + \mathbf{i}\tan\left(\tfrac{\alpha}{2}\right)\right).$$

The incoming frame F_{2m+2} of the vertex γ_{m+1} is the outgoing frame F_{2m+1} at the vertex γ_m rotated about N_{2m} by the length of the connecting edge δ, i.e.,

$$F_{2m+2} = F_{2m+1}\left(1 + \mathbf{k}\tan\left(\tfrac{\delta}{2}\right)\right).$$

proof of Theorem 7.1. Treat a triangle with edge lengths a, b, c and internal angles α, β, γ as a 3–periodic discrete curve. Then $F_m = \pm F_{m+6}$ implies $F_{m+5}^{-1}F_m \in \mathbb{R} \oplus \mathbb{R}\mathbf{k}$.
The angles between the edges of the triangle are the outer angles, i.e., $\pi - \alpha$, etc. Using $\tan\frac{\pi-\alpha}{2} = \cot\frac{\alpha}{2}$ we get

$$\left(1 + \mathbf{i}\cot\tfrac{\beta}{2}\right)\left(1 + \mathbf{k}\tan\tfrac{a}{2}\right)\left(1 + \mathbf{i}\cot\tfrac{\gamma}{2}\right)\left(1 + \mathbf{k}\tan\tfrac{b}{2}\right)\left(1 + \mathbf{i}\cot\tfrac{\alpha}{2}\right)$$

for $F_{m+5}^{-1}F_m$. The vanishing of the $\mathbf{i}$–part is Napier's Analogy. $\square$

8. Spectral Parameter of a Lorentz Harmonic Map

8.1. **Spherical parallelograms.** Applying Napier's Analogy 7.1 to a spherical parallelogram with edges δ and $\tilde{\delta}$ and inner angles φ and $\tilde{\varphi}$ yields

$$\tan\tfrac{\varphi}{2}\tan\tfrac{\tilde{\varphi}}{2} = \frac{1+\tan\frac{\delta}{2}\tan\frac{\tilde{\delta}}{2}}{1-\tan\frac{\delta}{2}\tan\frac{\tilde{\delta}}{2}}$$

8.2. **Associated Family and spectral parameter.** So given a discrete Lorentz harmonic map changing the side lengths δ and $\tilde{\delta}$ such that the value of $\tan\frac{\delta}{2}\tan\frac{\tilde{\delta}}{2}$ does not change one gets a new Lorentz harmonic map with these new side lengths and the same angles. This amounts to scaling by λ and $\frac{1}{\lambda}$ in the coordinate $\tan\frac{\delta}{2}$, which is stereographic projection of $e^{\mathbf{i}\delta}$. This family of Lorentz harmonic maps is called the *associated family* of the given Lorentz harmonic map and the parameter λ is called the *spectral parameter*.

9. Discrete Pendulum equation

9.1. **sine–Gordon equation.** A smooth K–surface is completely determined by the angle between its asymptotic lines φ and its constant Gauss curvature $K < 0$. The Gauss–Codazzi equations for an asymptotic line Chebyshev parametrization reduce to the sine–Gordon equation

$$\varphi_{uv} = -K\sin\varphi.$$

9.2. **Pendulum equation.** The sine–Gordon equation in xt–coordinates is $\varphi_{xx} - \varphi_{tt} = -4K\sin\varphi$. For surfaces of revolution the angle φ is independent of x and one gets the pendulum equation

$$\varphi_{tt} = 4K\sin\varphi.$$

9.3. **Discrete pendulum equation.** We may thus derive a discrete pendulum equation from discrete K–surfaces. The angles between the asymptotic lines are equal to the angles in the parallelogram of the Gauss map of the K–surface. The angles at a vertex of the Gauss map of a discrete K–surface satisfy

$$\varphi_u + \varphi_d + \varphi_r + \varphi_l = 2\pi.$$

Because the surface and its Gauss map are rotationally symmetric along the x–coordinate $\varphi_l = \varphi_r$. Writing indices for the t–coordinate only $\varphi_d = \varphi_{n-1}$, $\varphi_u = \varphi_{n+1}$, and writing $\tilde{\varphi}_n = \varphi_l = \varphi_r$ one gets

$$\varphi_{n+1} - 2\varphi_n + \varphi_{n-1} = -2(\varphi_n + \tilde{\varphi}_n)$$

modulo 2π. Since the Gauss map is Lorentz harmonic the quadrilateral with inner angles φ_n and $\tilde{\varphi}_n$ are parallelograms. Thus the equation derived in 8.1 holds.

Writing $K = \tan\frac{\delta}{2}\tan\frac{\tilde\delta}{2}$ this yields

$$\tan\frac{\varphi_n+\tilde\varphi_n}{2} = \frac{\tan\frac{\varphi_n}{2}+\tan\frac{\tilde\varphi_n}{2}}{1-\tan\frac{\varphi_n}{2}\tan\frac{\tilde\varphi_n}{2}} = \frac{\tan\frac{\varphi_n}{2}+\frac{1+K}{1-K}(\tan\frac{\varphi_n}{2})^{-1}}{1-\frac{1+K}{1-K}}$$
$$= \frac{(1-K)\sin^2\frac{\varphi_n}{2}+(1+K)\cos^2\frac{\varphi_n}{2}}{-2K\sin\frac{\varphi_n}{2}\cos\frac{\varphi_n}{2}} = \frac{1+K\cos\varphi_n}{-K\sin\varphi_n}$$

Discrete K–surfaces thus yield the following *discrete pendulum equation*

$$\text{(DP)} \qquad \varphi_{n+1}-2\varphi_n+\varphi_{n-1} = -4\arg(1+Ke^{\mathbf{i}\varphi_n})$$

This is similar to the discretization of the pendulum equation one gets by the *Verlet–method*:

$$\text{(VP)} \qquad \varphi_{n+1}-2\varphi_n+\varphi_{n-1} = -4K\sin(\varphi_n).$$

Both methods yield *symplectic integrators*. That means the first order transformation

$$T\colon \mathbb{R}^2\to\mathbb{R}^2, \qquad T(\varphi_{n-1},\varphi_n) = (\varphi_n,\varphi_{n+1})$$

induced in the phase space satisfies $\det(T') = 1$, or otherwise said, T leaves the symplectic form det on $\mathbb{R}^2$ invariant. This holds, because both are of the form $T(x,y) = (y, h(y)-x)$.
In contrast to the Verlet–integrator (VP) the integrator (DP) obtained from discrete K–surfaces is integrable in the sense that it posseses a constant of the motion, namely the monodromy of the Gauss map along the x–axis. This is a time independent rotation, because the rotational symmetry of the initial condition is preserved by the geometric construction (completion of spherical parallelograms).

9.4. **Theorem.** *The function*

$$\text{(9.1)} \qquad H(\varphi_{n-1},\varphi_n) = \left(\cos\left(\tfrac{\varphi_n}{2}-\tfrac{\varphi_{n-1}}{2}\right)+K\cos\left(\tfrac{\varphi_n}{2}+\tfrac{\varphi_{n-1}}{2}\right)\right)^2$$

is constant under the evolution of the discrete pendulum equation (DP), *i.e.,*

$$H\circ T = H, \quad \textit{where} \quad T\begin{pmatrix}\varphi_{n-1}\\ \varphi_n\end{pmatrix} = \begin{pmatrix}\varphi_n\\ 2\varphi_n-4\arg(1+Ke^{i\varphi_n})-\varphi_{n-1}\end{pmatrix}$$

Proof. The intrinsic geometric data of the Gauss map (the angles φ_n and the sidelengths δ and $\tilde\delta$ of the spherical parallelograms) of a surface of revolution is 1–periodic in x–direction. The monodromy along the x–axis is a rotation. This rotation is, in the sense of Theorem 4.8, represented by the quaternion

$$\lambda = \left(1+\mathbf{k}\tan\tfrac{\delta}{2}\right)\left(1+\mathbf{i}\cot\tfrac{\varphi_{n-1}}{2}\right)\left(1+\mathbf{k}\tan\tfrac{\tilde\delta}{2}\right)\left(1+\mathbf{i}\cot\tfrac{\varphi_n}{2}\right)$$

This can be obtained with the methods introduced in the proof of Napier's Analogy, Theorem 7.1.

Let ω denote the angle of rotation represented by λ, then one may check that

$$H(\varphi_{n-1}, \varphi_n) = (1 + \tan^2 \tfrac{\delta}{2})(1 + \tan^2 \tfrac{\tilde{\delta}}{2})(\tan \tfrac{\omega}{2} + 1)^{-1}.$$

□

The evolution T of the Verlet–integrator of the pendulum equation (VP) is the so called standard map of chaos theory, which should be interpreted in the context of the KAM–theorem as a perturbation of the integrable discrete pendulum equation (DP).
Figures 1 and 2 show (for $K = .04$ and $K = .3$, respectively) orbits in phase space obtained by the Runge–Kutta method, the Verlet–integrator (VP), and the integrable discrete pendulum equation (DP). The integrability of (DP) expressed in Theorem 9.4 implies that its orbits have no choice but to lie on the level sets of the constant of the motion H.

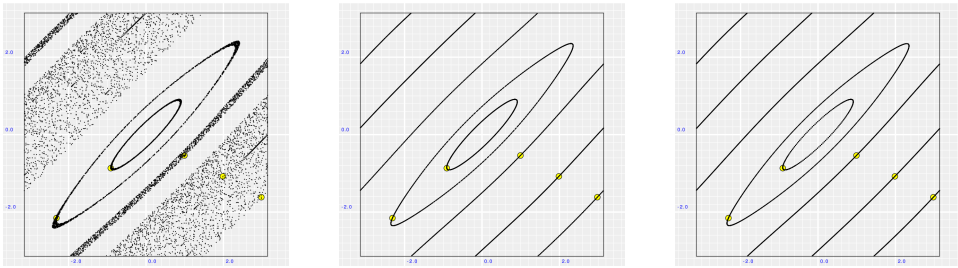

FIGURE 1. Phase space of the pendulum equation (Runge–Kutta, VP, DP)

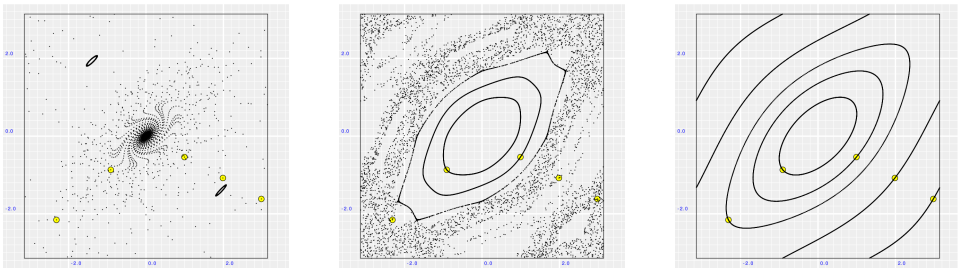

FIGURE 2. Phase space of the pendulum equation (Runge–Kutta, VP, DP)

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