Truncated moment problems, positive linear functionals, and finite algebraic varieties arising from cubic column relations

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Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{|i|\leq 2n}$ denote a d-dimensional real multisequence, let K denote a closed subset of $\mathbb{R}^d$, and let $\mathcal{P}_{2n} := \{p \in \mathbb{R}[x_1, \ldots, x_d] : \deg\ p \leq 2n\}$. Corresponding to β, the *Riesz functional* $L \equiv L_\beta : \mathcal{P}_{2n} \longrightarrow \mathbb{R}$ is defined by $L(\sum a_i x^i) := \sum a_i \beta_i$. We say that L is *K-positive* if whenever $p \in \mathcal{P}_{2n}$ and $p|_K \geq 0$, then $L(p) \geq 0$. In joint work with L.A. Fialkow, we prove that β admits a K-representing measure if and only if L_β admits a K-positive linear extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$. This provides a generalization (from the full moment problem to the truncated moment problem) of the Riesz-Haviland Theorem.

Truncated moment problems (TMP) as above for which the support of a representing measure is required to lie inside a closed set K are called truncated K-moment problems (TKMP). In case K is a semi-algebraic set determined by polynomials q_1,...,q_m, the study of TKMP is dual to determining whether a polynomial nonnegative on K belongs to the positive cone consisting of polynomials of degree at most $2n$ which can be expressed as sums of squares, and of squares multiplied by one or more distinct q_i's. Thus, our results also show that a semialgebraic set solves the truncated moment problem in terms of natural degree-bounded positivity conditions if and only if each polynomial strictly positive on that set admits a degree-bounded weighted sum-of-squares representation.

For the multisequence β to have a representing measure μ it is necessary for the associated moment matrix $M(n)$ to be positive semidefinite, and for the algebraic variety associated to β, V_β, to satisfy rank $M(n) \leq$ card V_β as well as the following *consistency condition*: if a polynomial $p(x) \equiv \sum_{|i|\leq 2n} a_i x^i$ vanishes on V_β, then $p(\beta) := \sum_{|i|\leq 2n} a_i \beta_i = 0$. In joint work with L.A. Fialkow and H.M. Möller, we proved that for the extremal case (rank $M(n) = \text{card } V_\beta$), positivity and consistency are sufficient for the existence of a (unique, rank $M(n)$-atomic) representing measure.

In recent joint work with S. Yoo we have considered cubic column relations in $M(3)$ of the form (in complex notation) $Z^3 = itZ + u\bar{Z}$, where u and t are real numbers. For (u,t) in the interior of a real cone, we prove that the algebraic variety V_β consists of exactly 7 points; we can then apply the above mentioned solution of the extremal moment problem to obtain a necessary and sufficient condition for the existence of a representing measure. This requires a new representation theorem for sextic polynomials in Z and $\bar{Z}$ which vanish in the 7-point set V_β.

The talk is based in part on joint work with L.A. Fialkow, with H.M. Möeller and with S. Yoo.