# Metrics and Geodesics in Control and Identification of Shapes and Geometries

M.C. Delfour

## 1 Courant Metrics

A natural way to construct a family of variable domains is to consider the images of a fixed subset of $\mathbf{R}^N$ by some family of transformations of $\mathbf{R}^N$. The structure and the topology of the images can be specified via the natural algebraic and topological structures of the space of transformations or equivalence classes of transformations for which the full power of function analytic methods is available. There are many ways to do that and specific constructions and choices are very much problem dependent.

In 1972 [6] introduced complete metric topologies on a family of domains of class $C^k$ that are the images of a fixed open domain (locally a $C^k$-epigraph) through a family of $C^k$-diffeomorphisms of $\mathbf{R}^N$. There the natural underlying algebraic structure is the *group structure* of the composition of transformations with the identity transformation as the neutral element. Her analysis culminates with the construction of a complete metric on the quotient of the group by an appropriate closed subgroup of transformations leaving the fixed subset unaltered. She called it the *Courant metric*. She introduced this terminology because it is proved in the book of CourantHilbert [3, p. 420], that the $n$-th eigenvalue of the Laplace operator depends continuously on the domain $\Omega$, where $\Omega = (I + f)\Omega_0$ is the image of a fixed domain $\Omega_0$ by $I + f$ and $f$ is a smooth mapping (cf. [7]). But there is no notion of a metric in that book. Her constructions naturally extend to other families of transformations of $\mathbf{R}^N$ or of fixed hold-alls $D$.

In this paper we first extend her generic constructions associated with the space $C_0(\mathbf{R}^N, \mathbf{R}^N)$ of mappings from $\mathbf{R}^N$ into $\mathbf{R}^N$ to a larger family of Banach spaces of mappings such as $C^k(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, $C^{k,1}(\overline{\mathbf{R}^N}, \mathbf{R}^N)$, or $\mathcal{B}^{\|}(\mathbf{R}^{\mathcal{N}}, \mathbf{R}^{\mathcal{N}})$ (cf.

Delfour and Zolésio [5]), and beyond to Fréchet spaces such as $\mathcal{B}(\mathbf{R}^{\mathcal{N}}, \mathbf{R}^{\mathcal{N}})$ or $C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$ of infinitely continuously differentiables mappings. We emphasize the *geodesic character* of the construction of the metric and its interpretation as trajectories of bounded variation on the group.

The next step in the construction is the choice of the closed subgroup of transformations of $\mathbf{R}^N$ that is very much problem dependent. Originally, it was chosen as the set of transformations that leave the underlying set or pattern unaltered. However, in some applications, it could be unaltered up to a translation, a rotation, or a flip. The underlying set or pattern can be a closed set or an open crack free set. This includes closed submanifolds of $\mathbf{R}^N$. It is shown that, as long as the subgroup is closed, we get a complete Courant metric on the quotient group. In this section we also characterize the tangent space to the group of transformations of $\mathbf{R}^N$ that leads to the Courant metric. It is an example of an infinite dimensional manifold where the tangent space is independent of the point.

Finally, we free the constructions from the framework of bounded continuously differentiable transformations to reach the spaces of all homeomorphisms or $C^k$-diffeomorphisms of $\mathbf{R}^N$ or an open subset $D$ of $\mathbf{R}^N$. Again it is shown that they are complete metric spaces. Hence, their quotient by a closed subgroup yields a Courant metric and a complete metric topology. With such larger spaces, it now becomes possible to consider subgroups involving not only translations but also isometries, symmetries, or flips in $\mathbf{R}^N$ or $D$.

# 2 Velocity Method

The quotient groups of transformations $\mathcal{F}(\Theta)/\mathcal{G}$ and their associated complete Courant metrics are neither linear nor convex spaces. We specialize the constructions to subspaces of transformations that are generated at time $t = 1$ by the flow of a *velocity field* over a generic time interval $[0, 1]$ with values in the tangent space $\Theta$. The main motivation is to introduce a notion of semiderivatives in the direction $\theta \in \Theta$ on such groups as well as a tractable criterion for continuity via $C^1$ or continuous paths in the quotient group endowed with the Courant matric.

This point of view was adopted by Zolésio [26, 27]] as early as 1973 and considerably expanded in his *thèse d'état* in 1979. One of his motivations was to solve a *shape differential equation* of the type $\mathcal{A}\, V(t) + G(\Omega_t(V)) = 0$,

$t > 0$, where $G$ is the *shape gradient* of a functional and $\mathcal{A}$ a duality operator[1]. At that time most people were using a simple perturbation of the identity to compute shape derivatives. The first comprehensive book systematically promoting the *velocity method* was published in 1992 by Sokolowski and Zolésio [12]. Structural theorems for the Eulerian Shape Derivative of smooth domains were first given in 1979 in [27] and generalized to nonsmooth domains in 1992 in [4]. The velocity point of view was also adopted in 1994 by R. Azencott [1] and his team (cf. for instance Trouvé [16, 14, 15]] in 1995 and 1998) to construct complete metrics and *geodesic paths* in spaces of diffeomorphisms generated by a velocity field with a broad spectrum of applications to imaging. The reader is referred to the forthcoming book of Younes [23] for a comprehensive exposition of this work and beyond to related papers such as [8], [9, 10], [24]. In view of the above motivations, this paper specializes the results to transformations generated by velocity fields. It also explores the connections between the constructions of Azencott and Micheletti that implicitly uses a notion of *geodesic path with discontinuities*.

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