The Jacobi matrices approach to Nevanlinna-Pick problems

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A modification of the well-known step-by-step process for solving Nevanlinna-Pick problems in the class of $\mathbf{R}_0$-functions gives rise to a linear pencil $H-\lambda J$, where H and J are Hermitian tridiagonal matrices. First, we show that J is a positive operator. Then we prove that the corresponding Nevanlinna-Pick problem has a unique solution if and only if a densely defined symmetric operator $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ is self-adjoint. Besides, some criteria for $J^{-\frac{1}{2}}HJ^{-\frac{1}{2}}$ to be self-adjoint are obtained. In the self-adjoint case, we get that multipoint diagonal Padé approximants converge to a unique solution of the Nevanlinna-Pick problem locally uniformly in $\mathbb{C}\setminus\mathbb{R}$.

To show the relation of our approach to the classical one we should notice that, roughly speaking, if all interpolation points tend to infinity simultaneously (that is, the Nevanlinna-Pick problem is approaching a moment problem) then the corresponding matrix J tends to the identity I elementwise. Thus, the proposed scheme extends the classical Jacobi matrix approach to moment problems and Padé approximation for $\mathbf{R}_0$-functions.

For the sake of completeness, let us recall that $\varphi\in\mathbf{R_0}$ if it has the following integral representation

$$\varphi(\lambda)=\int_{\mathbb{R}}\frac{d\sigma(t)}{t-\lambda},$$

where σ is a finite positive Borel measure (i.e. $\int_{\mathbb{R}}d\sigma(t)<\infty$).