The periodic decomposition problem for operator semigroups

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It was asked by I. Z. Ruzsa whether the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ can be written as the sum of n periodic functions with prescribed periods $a_1, \dots, a_n \in \mathbb{R}$. The answer to this question turns out to be dependent on the periods. One can reformulate and generalize this problem as follows. Consider the shift operators $T_a : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ defined by $(T_a f)(x) := f(x + a)$. Is it true that

$$\mathrm{id} \in \ker(T_{a_1} - I) + \cdots + \ker(T_{a_n} - I)?$$

(A function belongs to the right hand side if and only if it is a sum of n periodic functions.) Or, more generally, does the equality

$$\ker(T_{a_1} - I) \cdots (T_{a_n} - I) = \ker(T_{a_1} - I) + \cdots + \ker(T_{a_n} - I)$$

hold true? (Here $I : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is the identity mapping.) Note that for $n \geq 2$ the id belongs to the left hand side and that the inclusion "$\supseteq$" is trivial. Of course, this question is now meaningful for general linear operators T_i replacing the shifts T_{a_i}. We give some answers in the case when T_i are given commuting, bounded linear operators on a Banach space. Both algebraic and analytic properties of the semigroup generated by the operators $T_1, \dots, T_n$ play an important role here. The talk is partially based on a joint work with T. Keleti, V. Harangi and Sz. Gy. Révész.