Riemann-Hilbert problems on compact Riemann surfaces

G. Giorgadze

The local theory of ordinary differential equations in one dimensional case is the object of intensive study starting from Gauss to nowadays. In an unfinished paper "Zwei allgemeine Satze uber lineare Differentialgleichungen mit algebraischen Koeffizienten" B.Riemann formulated the problem of constructing ordinary differential equations with prescribed branching points and monodromy.

This problem appeared fundamental and very stimulating in the analytic theory of ordinary differential equations. Later on D.Hilbert made important contributions to the topic and included it in the list of his famous problems under number 21. For this reason it is often called Riemann-Hilbert problem or Hilbert's 21st problem. It should be noted that Riemann himself suggested a fruitful approach to this problem and, in particular, showed that it can be reduced to another problem which also appeared very interesting and useful. The problem consists in finding piecewise holomorphic matrix functions which satisfy certain boundary condition on the unit circle. This problem is called the problem of linear conjugation for analytic functions or simply Riemann problem.

In course of work over classification of linear differential equations with singular points, L.Fuchs introduced an important class of such equations, which in the modern literature are called Fuchsian equations. It turned out that the Riemann-Hilbert problem is especially interesting in the class of Fuchsian systems and research on this topic continues up to now.

Considerable progress in solving Riemann-Hilbert problem and linear conjugation problem was achieved by J.Plemelj [1].

Plemelj reduced linear conjugation problem with piecewise continuous coefficients to the case of continuous coefficients and applied this for solving Riemann-Hilbert problem. For a long time it was believed that he gave a complete solution of Riemann-Hilbert problem but later it turned out that there was a gap in his argument and eventually A.Bolibruch found a counterexample showing that Riemann-Hilbert problem is not always solvable in the class of Fuchsian systems [2].

It should be noted that an important contribution to the theory of Riemann-Hilbert problem was made by I.Lappo-Danilevsky [3], who developed a theory of functions of matrices and applied it to effective construction of solutions of certain Riemann-Hilbert problems. A big progress in the theory of linear conjugation problem was achieved by N.Muskhelishvili and N.Vekua (see [4] who, in particular, successfully used matrix calculus and introduced the so-called partial indices of matrix functions on smooth contours.

We consider analogical problems on compact Riemann surfaces (see [5]).

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References

[1] J.Plemelj. Problems in the sense of Riemann and Klein. Intersience Publishers. A division of J.Wiley & Sons Inc., New York, London, Sidney, 1964.

[2] A.A.Bolibruch. The Riemann-Hilbert problem, Russian Math.surveys, vol. 45 (1990), N 2, 1-47.

[3] I.Lappo-Danilevskii. Memoires sur la theorie des systemes des equations differentielles lineres, Chelsea, New-York,1953.

[4] N.I.Muskhelishvili. Singular integral equations. Noordhoff, Groningen,1953.

[5] G. Giorgadze. G-systems and holomorphic principal bundles on Riemann surfaces. J. Dyn. Contr. Syst. 8, N. 2, 2002, 245-291.