Sampling problems and oblique projections

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To sample a signal f means to obtain a sequence $\{f_n\}_{n\in\mathbb{N}}$ of instantaneous values of a particular signal characteristic, this sequence are called the samples of f. The classical sampling scheme is based on the Whittaker-Kotelnikov-Shannon theorem. Given a signal $f \in \mathcal{PW}$ (the Paley-Wiener space), the Whittaker-Kotelnikov-Shannon theorem establishes that it is possible to reconstruct the signal f from its samples $\{f_n\}_{n\in\mathbb{N}}$. When a signal $f \in L^2(\mathbb{R})$ does not belong to the Paley-Wiener space, a common strategy in signal processing applications is to apply a low pass filter (certain bounded linear operator) to the signal f obtaining a new signal g. Then, the filtered signal g is sampled giving the sequence $\{g_n\}_{n\in\mathbb{N}}$. Although, the signal recovered by the samples $\{g_n\}$ will not generally coincide with the original signal f, approximates it. In fact, the recovered signal is the best approximation, i.e., the orthogonal projection, of the original signal in $\mathcal{PW}$. A common way to represent the samples of a signal f, is by means of the inner product of f with some given vectors $\{v_n\}_n \in \mathbb{N}$ that spans a closed subspace $\mathcal{S}$, called the *sampling subspace*. By the other hand, given the the samples $\{f_n\}_{n\in\mathbb{N}}$, the reconstructed signal $\hat{f}$ is given by $\hat{f} = \sum_{n\in\mathbb{N}} f_n w_n$, where $\{w_n\}_{n\in\mathbb{N}}$ spans a closed subspace $\mathcal{R}$, called the *reconstruction subspace*.

In the classical sampling scheme the reconstruction and the sampling subspaces are assumed to be the same. In signal processing applications, this not always the case, and then it is not always possible to recover the best approximation of the original signal. Thus, different sampling techniques must be used. M. Unser and A. Aldroubi introduced the idea of consistent sampling, it means that the reconstructed signal $\hat{f}$ is not supposed to be the best approximation of the original signal, but f and $\hat{f}$ have the same samples. The main goal of this talk is to give an interpretation of the consistent sampling in terms of the notion of compatibility between a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and a positive semidefinite operator A acting on $\mathcal{H}$. This notion has a completely different origin. Z. Pasternak-Winiarski studied, for a fixed subspace $\mathcal{S}$, the analiticity of the map $A \to P_{A,\mathcal{S}}$ which associates to each positive invertible operator A the orthogonal projection onto $\mathcal{S}$ under the (equivalent) inner product $< \xi, \eta >_A = < A\xi, \eta >$, for $\xi, \eta \in \mathcal{H}$. Later E. Andruchow et al. presented a simplification of Pasternak-Winiarski's arguments and further geometrical results on the map $(A, \mathcal{S}) \to P_{A,\mathcal{S}}$. The notion of compatibility appears when A is allowed to be any positive semidefinite operator, not necessarily invertible (and even, a selfadjoint bounded linear operator). More precisely, A and $\mathcal{S}$ are said to be *compatible* if there exists a (bounded linear) projection Q with image $\mathcal{S}$ which satisfies $AQ = Q^*A$ (i.e., Q is Hermitian with respect to the semi-inner product $< .,. >_A$). Unlike what happens for invertible A's, it may happen that there is no such Q. These perturbations of the inner product occur quite frequently in applications. The consistent sampling scheme has not been studied as acting on perturbed inner spaces. But, studying the consistent sampling scheme in the semi-inner product spaces allows a simpler way to study some problems related with this notion.

The talk is based on a joint work with G. Corach.